Comment on definition of the naturality basis for photons.

Adam Szczepaniak, June 30 (somewhere over the Atlantic)

Let $[J, \tau_X](\lambda_\gamma, m, \lambda_N, \lambda'_N)$ be the GJ amplitude for production of a resonance of spin J, naturality, τ_X, and spin projection in the GJ frame, m with λ's referring to helicities of the photon, and the initial and final nucleon. For a $t - channel$ exchange, the upper (photon coupling) vertex is invariant under reflection in the production plane, the Π_y-parity, which leads to

$$[J, \tau_X](\lambda_\gamma, m, \lambda_N, \lambda'_N) = -\tau_X \tau_e (-)^m [J, \tau_X](-\lambda_\gamma, -m, \lambda_N, \lambda'_N) \quad (1)$$

Here τ_e is the naturality of the exchange. Similarly for the lower (nucleon) vertex

$$[J, \tau_X](\lambda_\gamma, m, \lambda_N, \lambda'_N) = \tau_e (-)^{\lambda'_N - \lambda_N} [J, \tau_X](\lambda_\gamma, m - \lambda_N, -\lambda'_N) \quad (2)$$

We want rewrite the spin density matrix

$$\rho^{[J\tau_X],[J'\tau'_X]}_{m,m'}(\lambda_\gamma, \lambda'_\gamma) = \sum_{\lambda_N, \lambda'_N} [J, \tau_X](\lambda_\gamma, m, \lambda_N, \lambda'_N)[J', \tau'_X]^*(\lambda'_\gamma, m', \lambda_N, \lambda'_N) \quad (3)$$

in terms of the eigenstates of the reflectivity operator. This will split up natural from unnatural exchanges. In the reflectivity basis (for both photon and produced resonance) the amplitudes are given by ($\epsilon_\gamma, \epsilon_X = \pm 1$),

$$[J, \tau_X]^{\epsilon_\gamma \epsilon_X}(|m|, \lambda_N, \lambda'_N) \equiv \theta(|m|)\,[[J, \tau_X]^{\epsilon_\gamma}(m, \lambda_N, \lambda'_N) + \epsilon_X \tau_X (-)^m [J, \tau_X]^{\epsilon_\gamma}(-m, \lambda_N, \lambda'_N)] \quad (4)$$

where $\theta(m) = 1/\sqrt{2}$ for $\neq 1$ and $= 1/2$ for $m = 0$ and

$$[J, \tau_X]^{\epsilon_\gamma}(m, \lambda_N, \lambda'_N) = \frac{1}{\sqrt{2}}\,[[J, \tau_X](\lambda_\gamma = +1, m, \lambda_N, \lambda'_N) - \epsilon_\gamma [J, \tau_X](\lambda_\gamma = -1, m, \lambda_N, \lambda'_N)] \quad (5)$$

For given λ_N, λ'_N there are $2 \times (2J_X + 1)$ amplitudes. Parity eliminates half of them and the remaining $2J_X + 1$ amplitudes are split into noninterfering amplitudes describing natural and unnatural exchange mechanisms. Using Eq. 1 one can show that these are given by

$$\begin{array}{ll}
\text{For } m \neq 0 & \\
[J, \tau_X]^{++}(|m|, \lambda_N, \lambda'_N) \neq 0, & \text{if } \tau_e = 1(N) \\
[J, \tau_X]^{--}(|m|, \lambda_N, \lambda'_N) \neq 0, & \text{if } \tau_e = 1(N) \\
[J, \tau_X]^{+-}(|m|, \lambda_N, \lambda'_N) \neq 0, & \text{if } \tau_e = -1(U) \\
[J, \tau_X]^{-+}(|m|, \lambda_N, \lambda'_N) \neq 0, & \text{if } \tau_e = -1(U) \\
\text{For } m = 0 & \\
[J, \tau_X]^{+\tau_X}(0, \lambda_N, \lambda'_N) \neq 0, & \text{if } \tau_X \tau_e = 1 \\
[J, \tau_X]^{-\tau_X}(0, \lambda_N, \lambda'_N) \neq 0, & \text{if } \tau_X \tau_e = -1
\end{array} \quad (6)$$

The amplitudes $[\cdots]^{+,\epsilon_X}(\cdots)$ correspond to photons linearly polarized in the production plane and $[\cdots]^{-,\epsilon_X}(\cdots)$ to photons polarized perpendicular to the production plane

$$\begin{aligned}
[\cdots]^{+,\epsilon_X}(\cdots) &= -[\cdots]^{\mathbf{x},\epsilon_X}(\cdots) \\
[\cdots]^{-,\epsilon_X}(\cdots) &= -i[\cdots]^{\mathbf{y},\epsilon_X}(\cdots)
\end{aligned} \tag{7}$$

Finally, $\epsilon_X, |m|$ label the reflectivity basis of the produced resonance

$$|[J_X,\tau_X],\epsilon_X,|m|\rangle = \theta(|m|)\left[|[J_X,\tau_X],|m|\rangle + \epsilon_X\tau_X(-)^m|[J_X,\tau,X],-|m|\rangle\right] \tag{8}$$

In terms of these amplitudes the density matrix is given by

$$\begin{aligned}
&\rho^{[J\tau_X],[J'\tau'_X]}_{\epsilon_X,|m|,\epsilon'_X,|m'|}(\alpha) = \\
&\sin(\alpha)^2\frac{(1+\tau_e\epsilon_X)}{2}\frac{(1+\tau_e\epsilon'_X)}{2}[J,\tau_X]^{+\epsilon_X}(|m|)\left([J',\tau'_X]^{+\epsilon'_X}(|m'|)\right)^* \\
&+\cos(\alpha)^2\frac{(1-\tau_e\epsilon_X)}{2}\frac{(1-\tau_e\epsilon'_X)}{2}[J,\tau_X]^{-\epsilon_X}(|m|)\left([J',\tau'_X]^{-\epsilon'_X}(|m'|)\right)^* \\
&+i\sin(\alpha)\cos(\alpha)\left[\frac{(1+\tau_e\epsilon_X)}{2}\frac{(1-\tau_e\epsilon'_X)}{2}[J,\tau_X]^{+\epsilon_X}(|m|)\left([J',\tau'_X]^{-\epsilon'_X}(|m'|)\right)^*\right. \\
&\left.-\frac{(1-\tau_e\epsilon_X)}{2}\frac{(1+\tau_e\epsilon'_X)}{2}[J,\tau_X]^{-\epsilon_X}(|m|)\left([J',\tau'_X]^{+\epsilon'_X}(|m'|)\right)^*\right]
\end{aligned} \tag{9}$$

Here α is the orientation of the photon polarization vector with respect to the direction perpendicular to the production plane (in the GJ frame). Sum over nucleon helicities is implicit *i.e*

$$\begin{aligned}
[J,\tau_X]^{++}(|m|)\left([J',\tau'_X]^{++}(|m|')\right) &= [J,\tau_X]^{++}(|m|,+\frac{1}{2},+\frac{1}{2})\left([J,\tau_X](|m|',+\frac{1}{2},+\frac{1}{2})\right)^* \\
&+[J,\tau_X]^{++}(m|,+\frac{1}{2},-\frac{1}{2})\left([J'\tau'_X]^{++}(|m|',+\frac{1}{2},-\frac{1}{2})\right)^*
\end{aligned} \tag{10}$$

0.1 Example a_1, a_2, π_1, π_2 waves

In this case the nonvanishing amplitudes are :

a_1 Natural exchange $[a_1]^{++}(|m|=1)$ and $[a_1]^{--}(|m|=1,0)$ Unnatural exchange $[a_1]^{+-}(|m|=1,0)$ and $[a_1]^{-+}(|m|=1)$

a_2 Natural exchange $[a_2]^{++}(|m|=2,1,0)$ and $[a_2]^{--}(|m|=2,1)$ Unnatural exchange $[a_2]^{+-}(|m|=2,1)$ and $[a_2]^{-+}(|m|=2,1,0)$

π_1 Natural exchange $[\pi_1]^{++}(|m|=1,0)$ and $[\pi_1]^{--}(|m|=1)$ Unnatural exchange $[\pi_1]^{+-}(|m|=1)$ and $[\pi_1]^{-+}(|m|=1,0)$

π_2 Natural exchange $[\pi_2]^{++}(|m|=2,1)$ and $[\pi_2]^{--}(|m|=2,1,0)$ Unnatural exchange $[\pi_2]^{+-}(|m|=2,1,0)$ and $[\pi_2]^{-+}(|m|=2,1)$

Here dependence on nucleon spin is implicit, one needs (in general) a spin-nonflip and a spin flip amplitude. However if nucleon polarization information is unavailable one will measure a coherent sum of the two (cf Eq. 10).

0.2 a_2 production via OPE

In this case

$$[a_2](\lambda_\gamma, m, \lambda_N, \lambda'_N) = A\delta_{\lambda_N,-\lambda'_N}\tau^3_{\lambda_\gamma,m} \tag{11}$$

(τ^3 is the Pauli matrix) ¿From Eq. 4 for the only nonvanishing amplitudes one gets

$$[a_2]^{+-}(\lambda_\gamma=1,\lambda_N,\lambda'_N) = [a_2]^{-+}(\lambda_\gamma=1,\lambda_N,\lambda'_N) = A\delta_{\lambda_N,-\lambda'_N} \tag{12}$$

i.e. only nucleon helicity flip contributes. The spin density matrix are then given by

$$\begin{aligned}
\rho^{[a_2],[a_2]}_{-,1,-,1}(\alpha) &= \sin(\alpha)^2|[a_2]^{+-}(\lambda_\gamma=1)|^2 = \sin(\alpha)^2|A|^2 \\
\rho^{[a_2],[a_2]}_{-,1,+,1}(\alpha) &= i\sin(\alpha)\cos(\alpha)[a_2]^{+-}\left([a_2]^{-+}\right)^* = i\sin(\alpha)\cos(\alpha)|A|^2 \\
\rho^{[a_2],[a_2]}_{+,1,-,1}(\alpha) &= -i\sin(\alpha)\cos(\alpha)[a_2]^{-+}\left([a_2]^{+-}\right)^* = -i\sin(\alpha)\cos(\alpha)|A|^2 \\
\rho^{[a_2],[a_2]}_{+,1,+,1}(\alpha) &= \cos(\alpha)^2|[a_2]^{-+}|^2 = \cos(\alpha)^2|A|^2
\end{aligned} \tag{13}$$

0.3 Case of fractional polarization

In the helicity basis the photon density matrix is given by

$$\rho_{\lambda_\gamma,\lambda'_\gamma} = \frac{1}{2}\left[I + P\vec{n}\cdot\vec{\sigma}\right]_{\lambda_\gamma,\lambda'_\gamma} \tag{14}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & e^{2i\alpha} \\ e^{-2i\alpha} & 1 \end{pmatrix} \tag{15}$$

where $0 < P < 1$ is the degree of linear polarization and

$$n = (\cos 2\alpha, -\sin 2\alpha, 0) \tag{16}$$

In the basis of linearly polarized photons :

$$
\begin{aligned}
|\lambda_\gamma = +\rangle &= -\frac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right) \\
|\lambda_\gamma = -\rangle &= \frac{1}{\sqrt{2}}\left(|x\rangle - i|y\rangle\right)
\end{aligned}
\tag{17}
$$

this gives $(i, j = x, ory)$

$$
\rho_{ij} = \frac{1}{2}\begin{pmatrix} 1 - P\cos 2\alpha & P\sin 2\alpha \\ P\sin 2\alpha & 1 + P\cos 2\alpha \end{pmatrix} \tag{18}
$$

This is diagonal in the basis given by two orthogonal vectors

$$
\begin{aligned}
|1\rangle &= \sin\alpha|x\rangle + \cos\alpha|y\rangle| \\
|2\rangle &= -\cos\alpha|x\rangle + \sin\alpha|y\rangle
\end{aligned}
\tag{19}
$$

corresponding to eigenvalues $(1+P)/2$ and $(1-P)/2$ respectively.

It s now straightforward to generalize resonance production spin density matrix $\rho^{[J\tau_X],[J'\tau'_X]}_{\epsilon_X,|m|,\epsilon'_X,|m'|}(\alpha)$ to the case when $P \neq 1$

$$
\begin{aligned}
&\rho^{[J\tau_X],[J'\tau'_X]}_{\epsilon_X,|m|,\epsilon'_X,|m'|}(\alpha, P) = \left(\frac{1+P}{2}\right)\Bigg[\sin(\alpha)^2 \frac{(1+\tau_e\epsilon_X)}{2}\frac{(1+\tau_e\epsilon'_X)}{2}[J,\tau_X]^{+\epsilon_X}(|m|)\left([J',\tau'_X]^{+\epsilon'_X}(|m'|)\right)^* \\
&+\cos(\alpha)^2 \frac{(1-\tau_e\epsilon_X)}{2}\frac{(1-\tau_e\epsilon'_X)}{2}[J,\tau_X]^{-\epsilon_X}(|m|)\left([J',\tau'_X]^{-\epsilon'_X}(|m'|)\right)^* \\
&+i\sin(\alpha)\cos(\alpha)\Bigg[\frac{(1+\tau_e\epsilon_X)}{2}\frac{(1-\tau_e\epsilon'_X)}{2}[J,\tau_X]^{+\epsilon_X}(|m|)\left([J',\tau'_X]^{-\epsilon'_X}(|m'|)\right)^* \\
&-\frac{(1-\tau_e\epsilon_X)}{2}\frac{(1+\tau_e\epsilon'_X)}{2}[J,\tau_X]^{-\epsilon_X}(|m|)\left([J',\tau'_X]^{+\epsilon'_X}(|m'|)\right)^*\Bigg]\Bigg] \\
&+\left(\frac{1-P}{2}\right)\Bigg[\cos(\alpha)^2 \frac{(1+\tau_e\epsilon_X)}{2}\frac{(1+\tau_e\epsilon'_X)}{2}[J,\tau_X]^{+\epsilon_X}(|m|)\left([J',\tau'_X]^{+\epsilon'_X}(|m'|)\right)^* \\
&+\sin(\alpha)^2 \frac{(1-\tau_e\epsilon_X)}{2}\frac{(1-\tau_e\epsilon'_X)}{2}[J,\tau_X]^{-\epsilon_X}(|m|)\left([J',\tau'_X]^{-\epsilon'_X}(|m'|)\right)^* \\
&-i\sin(\alpha)\cos(\alpha)\Bigg[\frac{(1+\tau_e\epsilon_X)}{2}\frac{(1-\tau_e\epsilon'_X)}{2}[J,\tau_X]^{+\epsilon_X}(|m|)\left([J',\tau'_X]^{-\epsilon'_X}(|m'|)\right)^* \\
&-\frac{(1-\tau_e\epsilon_X)}{2}\frac{(1+\tau_e\epsilon'_X)}{2}[J,\tau_X]^{-\epsilon_X}(|m|)\left([J',\tau'_X]^{+\epsilon'_X}(|m'|)\right)^*\Bigg]\Bigg]
\end{aligned}
\tag{20}
$$