BNL PREPRINT

BNL-QGS-02-0900
CERN 71-8

SPIN FORMALISMS

—Updated Version—

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March 2, 2004

* under contract number DE-AC02-98CH10886 with the U.S. Department of Energy

FOREWARD

One of the basic problems in the study of elementary particle physics is that of describing the states of a system consisting of several particles with spin. This report represents an attempt to present a coherent and comprehensive view of the various spin formalisms employed in the study of the elementary particles. Particular emphasis is given to the description of resonances decaying into two, three or more particles and the methods of determining the spin and parity of resonances with sequential decay modes.

Relativistic spin formalisms are based on the study of the inhomogeneous Lorentz group called the Poincaré group. This report, however, is not a systematic study of this group. It is our opinion that most of the features of the spin formalisms may be understood on a more elementary and intuitive level. Certainly, a deeper understanding of the subject is possible only from a careful study of the Poincaré group. Suffice it to say that the group possesses two invariants corresponding to the mass and the spin of a particle and that all possible states of a free particle with arbitrary mass and spin form the set of basis vectors for an irreducible representation of the group.

Our approach here is to start with the particle states at rest, which are the eigen-vectors corresponding to the standard representation of angular momentum, and then "boost" the eigenvectors to obtain states for relativistic particles with arbitrary momentum. If the boost operator corresponds to a *pure* Lorentz transformation, we obtain the canonical basis of state vectors which, in this report, we call the canonical states for brevity. On the other hand, a certain boost operator corresponding to a mixture of a pure Lorentz transformation and a rotation defines the helicity state vectors whose quantization axis is taken along the direction of the momentum. Of course, this approach precludes discussion of massless particles on the same footing. We may point out, however, that states of a massless particle can best be treated in the helicity basis, with the proviso that the helicity quantum number be restricted to positive or negative values of the spin. In this report, we deal exclusively with the problem of describing the hadronic states.

In Sections 1 to 4, we develop concurrently the canonical and helicity states for one- and two-particle systems. In Section 5 we discuss the partial-wave expansion of the scattering amplitude for two-body reactions and describe in detail the decay of a resonance into two particles with arbitrary spin. The treatment of a system consisting of three particles is given in the helicity basis in Section 6.

Section 7 is devoted to a study of the spin-parity analysis of two-step decay processes, in which each step proceeds via a pion emission. We give a formalism treating both baryon and boson resonances on an equal footing, and illustrate the method with a few simple but, in practice, important examples. In developing the formalism, we have endeavored to make a judicious choice of notation, in order to bring out the basic principles as simply as possible.

In the last two sections, Sections 8 and 9, we discuss the tensor formalism for arbitrary spin, the relativistic version of which is known as the Rarita-Schwinger formalism. In the case of integral spin, the starting point is the polarization vectors or the spin-1 wave functions embedded in four-momentum space. The boost operators in this case correspond to the familiar four-vector representation of the Lorentz transformations. In the case of half-integral spin, we start with four-component Dirac formalism for spin-$\frac{1}{2}$ states; the boost operators here correspond to the 4×4 non-unitary representation of the homogeneous Lorentz group. We derive explicit expressions for the wave functions for a few lower spin values and exhibit the form of the corresponding spin projection operators. Particular emphasis is given, through a series of examples, to elucidating the connection between the formalism of Rarita and Schwinger and that of the non-relativistic spin tensors developed by Zemach, as well as the relationship between these and the helicity formalism of Section 4.

It is in the spirit "Best equipped is he who can wield all tools available" that we have attempted to present here a coherent and unified study of the spin formalisms that are frequently employed in the study of resonant states. This report, however, is not an exhaustive treatise on the subject; rather, it represents an elementary, but reasonably self-contained account of the basic underlying principles and simple applications. We give below a list of general references, either as a supplementary material for the subjects treated briefly in this report, or as a source of alternative approaches to the methods developed here.

On the subject of angular momentum and related topics such as the Clebsch-Gordan coefficients and the Wigner D-functions, the reader is referred to Messiah [1], Rose [2], and Edmonds [3]. A thorough account of the irreducible unitary representation of the Poincaré group is given in Werle [4] and in Halpern [5]; a more concise exposition of the subject may be found in Gasiorowicz [6], Wick [7], Froissart and Omnès [8], and Moussa and Stora [9]. For a good treatment of the resonance decays covered in Sections 5 to 7, the reader is referred to Jackson [10]. Pilkuhn [11] gives a brief account of the spin tensors discussed in Sections 8 and 9. A systematic study of the relativistic spin states in a direction not covered in this report has been made by Weinberg [12] who has used the finite dimensional states of arbitrary spin. Some of the notations we have used are, however, those of Weinberg. We have not attempted to give a complete list of references on the subject of spin formalisms; the reader is referred to Jackson [10] for a more extensive list of references. See also Tripp [13] for a comprehensive survey on the methods of spin-parity analysis which have been applied to the study of resonant states.

The author is indebted to Drs. M. Jacob and P. Auvil for several enlightening discussions and to Drs. Reucroft and V. Chaloupka for careful reading of the draft. The author wishes to thank Dr. L. Montanet for his encouragement and support and the Track Chambers Division on CERN for its warm hospitality.

FOREWARD TO THE UPDATED VERSION

This updated version has been produced by Ms Fern Simes/BNL. I am very much indebted for her patience and her extraordinary skill in converting the orginal text into a LaTeX file. I acknowledge with pleasure the expert advice of Dr. Frank Paige/BNL on the interface problems relating to TeX and LaTeX. I am indebted to Dr. Norman McCubbin/Rutherford for sending me his suggested corrections and the typographical errors that he has found in Sections 1 through 5. This 'Second Version' incorporates, in addition, all the corrections that I have found necessary over the intervening years and the typographical errors that I have come across.

There has been a time span of some 32 years, since I produced the original version while I worked at CERN on leave-of-absence from BNL. I have since then come across many items in the report which I would have treated differently today. Instead of rewriting the report from scratch, I have decided to give a series of references, which I have found useful or which represent an advance in the topics covered in the original version.

There are two textbooks and/or monographs which I consult often. They are:

- A. D. Martin and T. D. Spearman, '*Elementary Particle Theory,*'
  North-Holland Publishing Co., Amsterdam (1970).
- Steven Weinberg, '*The Quantum Theory of Fields,*'
  Volume I (Cambridge University Press, Cambridge, 1995), Chapter 2.

Another recent monograph which the reader may find useful is

- Elliot Leader, '*Spin in Particle Physics,*'
  (Cambridge University Press, Cambridge, 2001),

There are two important references on the Poincarè group which should be mentioned:

- A. J. Macfarlane, J. Math. Phys. **4**, 490 (1963).
- A. McKerrell, NC **34**, 1289 (1964).

The first work treats the topic of the intrinsic spin of a two-particle system in terms of the generators of the Poincarè group. The second work introduces, for the first time, the concept of the relativistic orbital angular momenta, again starting from the generators of the Poincarè group.

The Zemach formalism, used widely in hadron spectroscopy, is inherently non-relativistic. A proper treatment requires introduction of the Lorentz factors $\gamma = E/w$, where w is the mass of a daughter state and E is its energy in the parent rest frame. There are some recent references which deal with this topic:

- S. U. Chung, Pys. Rev. D **48**, 1225 (1995).
- S. U. Chung, Pys. Rev. D **57**, 431 (1998).
- V. Filippini, A. Fonatana and A. Rotondi, Pys. Rev. D **51**, 2247 (1995).

The reader may consult the second reference above for a formula for the general wave function, corresponding to $|JM\rangle$ where J is an integer, which are constructed out of the familiar polarization 4-vectors. An independent derivation of the same formula has recently been carried out by

- S. Huang, T. Ruan, N. Wu and Z. Zheng,
  Eur. Phys. J. C **26**, 609 (2003).

They have also worked out an equivalent formula for half-integer spins.

One of the most important decay channels for a study of mesons is that involving two pseudoscalar particles. Such a decay system is described by the familiar sperical harmonics, but the analysis is complicated by the amibiguities in the partial-wave amplitudes. A general method of dealing with such a problem can be found in

- S. U. Chung, Pys. Rev. D **56**, 7299 (1997).

When a meson decays into two identical particles, the ensuing amibituity problem requires introduction of a new polynomial; see the Appendix B for a derivation of the polynomial.

The reader will note that the references cited here are *not* intended to be exhaustive. It is intended merely a guide—somehwat personal—for further reading.

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Contents

1 One-Particle States at Rest

States of a single particle at rest (mass $w > 0$) may be denoted by $|jm\rangle$, where j is the spin and m the z-component of the spin. The states $|jm\rangle$ are the canonical basis vectors by which the angular momentum operators are represented in the standard way. The procedure for representing the angular momentum operators is a familiar one from non-relativistic quantum mechanics[2]. We merely list here the main properties for later reference. Since the angular momentum operators are the infinitesimal generators of the rotation operator, the spin of a particle characterizes how the particle at rest transforms under spatial rotations.

Let us denote the three components of the angular momentum operator by J_x, J_y, and J_z (or J_1, J_2, and J_3). They are Hermitian operators satisfying the following commutation relations:

$$[J_i, J_j] = i\,\epsilon_{ijk}\,J_k\ , \tag{1.1}$$

where i, j, and k run from 1 to 3. The operators J_i act on the canonical basis vectors $|jm\rangle$ as follows:

$$\begin{aligned} J^2|jm\rangle &= j(j+1)|jm\rangle \\ J_z|jm\rangle &= m|jm\rangle \\ J_\pm|jm\rangle &= [(j\mp m)(j\pm m+1)]^{\frac{1}{2}}|jm\pm 1\rangle\ , \end{aligned} \tag{1.2}$$

where $J^2 = J_x^2 + J_y^2 + J_z^2$ and $J_\pm = J_x \pm iJ_y$. The states $|jm\rangle$ are normalized in the standard way and satisfy the completeness relation:

$$\begin{gathered} \langle j'm'|jm\rangle = \delta_{j'j}\,\delta_{m'm} \\ \sum_{jm} |jm\rangle\langle jm| = I\ , \end{gathered} \tag{1.3}$$

where I denotes the identity operator.

A finite rotation of a physical system (with respect to fixed coordinate axes) may be denoted by $R(\alpha,\beta,\gamma)$, where (α,β,γ) are the standard Euler angles. To each R, there corresponds a unitary operator $U[R]$, which acts on the states $|jm\rangle$, and preserves the multiplication law:

$$U[R_2\,R_1] = U[R_2]\,U[R_1]\ .$$

Now the unitary operator representing a rotation $R(\alpha,\beta,\gamma)$ may be written

$$U[R(\alpha,\beta,\gamma)] = e^{-i\alpha J_z}\,e^{-i\beta J_y}\,e^{-i\gamma J_z} \tag{1.4}$$

corresponding to the rotation of a physical system (active rotation!) by γ around the z-axis, β around the y-axis, and finally by α around the z-axis, with respect to a fixed (x,y,z) coordinate system. Then rotation of a state $|jm\rangle$ is given by

$$U[R(\alpha,\beta,\gamma)]\,|jm\rangle = \sum_{m'} |jm'\rangle\,D^j_{m'm}(\alpha,\beta,\gamma)\ , \tag{1.5}$$

where $D^j_{m'm}$ is the standard rotation matrix as given by Rose[2]:

$$\begin{aligned} D^j_{m'm}(R) = D^j_{m'm}(\alpha,\beta,\gamma) &= \langle jm'|\,U[R(\alpha,\beta,\gamma)]\,|jm\rangle \\ &= e^{-im'\alpha}\,d^j_{m'm}(\beta)\,e^{-im\gamma} \end{aligned} \tag{1.6}$$

and

$$d^j_{m'm}(\beta) = \langle jm'|e^{-i\beta J_y}|jm\rangle \ . \tag{1.7}$$

In Appendix A some useful formulae involving $D^j_{m'm}$ and $d^j_{m'm}$ are listed.

2 Relativistic One-Particle States

Relativistic one-particle states with momentum $\vec{p}$ may be obtained by applying on the states $|jm\rangle$ a unitary operator which represents a Lorentz transformation that takes a particle at rest to a particle of momentum $\vec{p}$. There are two distinct ways of doing this, leading to canonical and helicity descriptions of relativistic free particle states.

Let us first consider an arbitrary four-momentum p^μ defined by

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p}\,) \ . \tag{2.1}$$

With the metric tensor given by

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \tag{2.2}$$

we can also define a four-momentum with lower indices:

$$p_\mu = g_{\mu\nu}\, p^\nu = (E, -\vec{p}\,) \ . \tag{2.3}$$

The proper homogeneous orthochronous Lorentz transformation takes the four-momentum p^μ into p'^μ as follows:

$$p'^{\,\mu} = \Lambda^\mu{}_\nu\, p^\nu \ , \tag{2.4}$$

where $\Lambda^\mu{}_\nu$ is the Lorentz transformation matrix defined by

$$g_{\alpha\beta}\, \Lambda^\alpha{}_\mu\, \Lambda^\beta{}_\nu = g_{\mu\nu}, \qquad \det\Lambda = 1, \qquad \Lambda^0{}_0 > 0 \ . \tag{2.5}$$

The Lorentz transformation given by $\Lambda^\mu{}_\nu$ includes, in general, rotations as well as the pure Lorentz transformations. Let us denote by $L^\mu{}_\nu(\vec{\beta}\,)$ a pure time-like Lorentz transformation, where $\vec{\beta}$ is the velocity of the transformation. Of particular importance is the pure Lorentz transformation along the z-axis, denoted by $L_z(\beta)$:

$$L_z(\beta) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} \tag{2.6}$$

where $\beta = \tanh\alpha$.

In terms of $L_z(\beta)$, it is easy to define a pure Lorentz transformation along an arbitrary direction $\vec{\beta}$:

$$L(\vec{\beta}\,) = R(\phi,\theta,0) L_z(\beta) R^{-1}(\phi,\theta,0) \ , \tag{2.7}$$

where $R(\phi, \theta, 0)$ is the rotation which takes the z-axis into the direction of $\vec{\beta}$ with spherical angles (θ, ϕ):

$$\hat{\beta} = R(\phi, \theta, 0)\hat{z} \ . \tag{2.8}$$

The relation (2.7) is an obvious one, but the reader can easily check for a special case with $\phi = 0$:

$$R(\phi, \theta, 0) = \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix}, \quad R_{ij} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{2.9}$$

Now the action of an arbitrary Lorentz transformation Λ on relativistic particle states may be represented by a unitary operator $U[\Lambda]$. The operator preserves the multiplication law, called the group property:

$$U[\Lambda_2\Lambda_1] = U[\Lambda_2]\, U[\Lambda_1] \ . \tag{2.10}$$

Let us denote by $L(\vec{p}\ )$ the "boost" which takes a particle with mass $w > 0$ from rest to momentum $\vec{p}$ and the corresponding unitary operator acting on the particle states by $U[L(\vec{p}\ )]$:

$$U[L(\vec{p}\ )] = e^{-i\alpha\hat{p}\cdot\vec{K}} \ , \tag{2.11}$$

where $\tanh\alpha = p/E$, $\sinh\alpha = p/w$, and $\cosh\alpha = E/w$.

In analogy to Eq. (1.4), a boost operator defines a Hermitian vector operator $\vec{K}$, and the components K_i are then the infinitesimal generators of "boosts". The three components K_i together with J_i form the six infinitesimal generators of the homogeneous Lorentz group, and they satisfy definite commutation relations among them. We do not list the relations here, for they are not needed for our purposes. The interested reader is referred to Werle [4].

From the relation (2.7) and the group property (2.10), one obtains

$$U[L(\vec{p}\ )] = U[\mathring{R}(\phi, \theta, 0)]U[L_z(p)]U^{-1}[\mathring{R}(\phi, \theta, 0)] \ , \tag{2.12}$$

where the rotation $\mathring{R}$ takes the z-axis into the direction of $\vec{p}$ with spherical angles (θ, ϕ):

$$\hat{p} = \mathring{R}(\phi, \theta, 0)\hat{z} \ . \tag{2.13}$$

We are now ready to define the "standard" or canonical state describing a single particle with spin j and momentum $\vec{p}$:

$$\begin{aligned} |\vec{p}, jm\rangle &= |\phi, \theta, p, jm\rangle = U[L(\vec{p}\ )]\, |jm\rangle \\ &= U[\mathring{R}(\phi, \theta, 0)]\, U[L_z(p)]\, U^{-1}[\mathring{R}(\phi, \theta, 0)]\, |jm\rangle \ , \end{aligned} \tag{2.14}$$

where $|jm\rangle$ is the particle state at rest as defined in the previous section. We emphasize that the z-component of spin m is measured in the rest frame of the particle and *not* in the frame where the particle has momentum $\vec{p}$.

The advantage of the canonical state as defined in Eq. (2.14) is that the state transforms formally under rotation in the same way as the "rest-state" $|jm\rangle$:

$$\begin{aligned} U[R]\,|\vec{p},jm\rangle &= U[R\overset{\circ}{R}]\,U[L_z(p)]\,U^{-1}[R\overset{\circ}{R}]\,U[R]\,|jm\rangle \\ &= \sum_{m'} D^j_{m'm}(R)\,|R\,\vec{p},jm'\rangle\;, \end{aligned} \tag{2.15}$$

where one has used Eq. (1.5). It is clear from the relation (2.15) that one may take over all the non-relativistic spin formalisms and apply them to situations involving relativistic particles with spin. One ought to remember, however, that the z-component of spin is defined only in the particle rest frame obtained from the frame where the particle has momentum $\vec{p}$ via the pure Lorentz transformation $L^{-1}(\vec{p}\,)$ as given in Eq. (2.7) [see Fig. 1.1(a)].

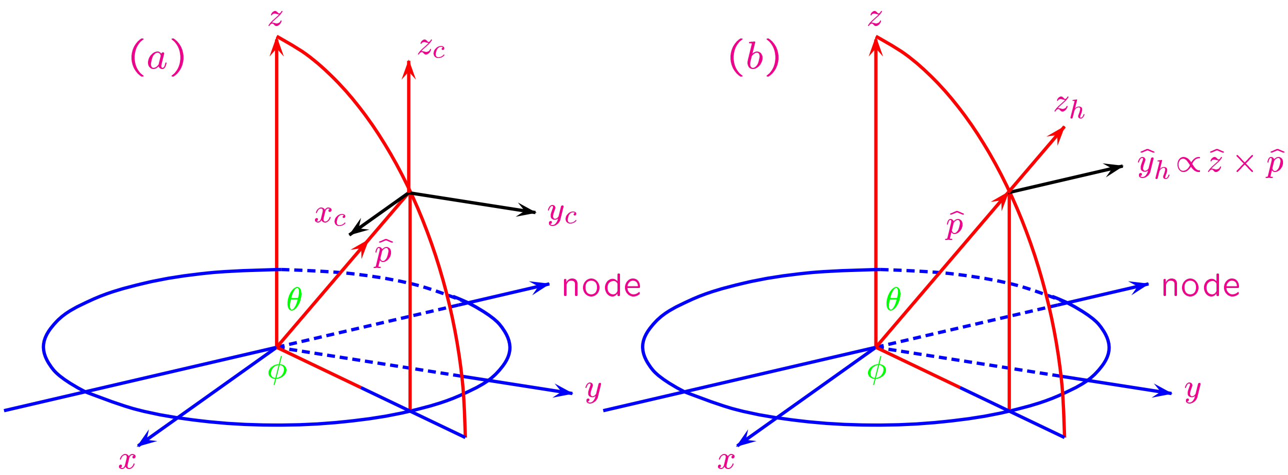

Figure 1.1: The orientation of the coordinate systems associated with a particle at rest in the (a) canonical ($\hat{x}_c$, $\hat{y}_c$, $\hat{z}_c$), and (b) helicity description ($\hat{x}_h = \hat{y}_h \times \hat{z}_h$, $\hat{y}_h \propto \hat{z} \times \hat{p}$, $\hat{z}_h = \hat{p}$).

Next, we shall define the helicity state describing a single particle with spin j and momentum $\vec{p}$ [see Fig. 1.1(b)]:

$$\begin{aligned} |\vec{p},j\lambda\rangle = |\phi,\theta,p,j\lambda\rangle &= U[L(\vec{p}\,)]\,U[\overset{\circ}{R}(\phi,\theta,0)]\,|j\lambda\rangle \\ &= U[\overset{\circ}{R}(\phi,\theta,0)]\,U[L_z(p)]\,|j\lambda\rangle\;. \end{aligned} \tag{2.16}$$

Helicity states may be defined in two different ways. One may first rotate the rest state $|j\lambda\rangle$ by $\overset{\circ}{R}$, so that the quantization axis is along the $\vec{p}$ direction and then boost the system along $\vec{p}$ to obtain the helicity state $|\vec{p},j\lambda\rangle$. Or, equivalently, one may first boost the rest state $|j\lambda\rangle$ along the z-axis and then rotate the system to obtain the state $|\vec{p},j\lambda\rangle$. That these two different definitions of helicity state are equivalent is obvious from the relation (2.12).

One sees that, by definition, the helicity quantum number λ is the component of the spin along the momentum $\vec{p}$, and as such it is a rotationally invariant quantity, simply because the quantization axis itself rotates with the system under rotation. This fact may be seen easily from the definition (2.16):

$$\begin{aligned} U[R]\,|\vec{p}, j\lambda\rangle &= U[R\mathring{R}]\,U[L_z]\,|j\lambda\rangle \\ &= |R\,\vec{p}, j\lambda\rangle \ . \end{aligned} \tag{2.17}$$

In addition, the helicity λ remains invariant under pure Lorentz transformation that takes $\vec{p}$ into $\vec{p}\,'$, which is parallel to $\vec{p}$. Then, the invariance of λ under L' may be seen by

$$\begin{aligned} U[L']\,|\vec{p}, j\lambda\rangle &= U[L']U[L(\vec{p}\,)]U[\mathring{R}]\,|j\lambda\rangle \\ &= U[L(\vec{p}\,')]U[\mathring{R}]\,|j\lambda\rangle \\ &= |\vec{p}\,', j\lambda\rangle \ . \end{aligned} \tag{2.18}$$

There is a simple connection between the canonical and helicity descriptions. From the definitions (2.14) and (2.16), on finds easily that

$$\begin{aligned} |\vec{p}, j\lambda\rangle &= U[\mathring{R}]U[L_z]U^{-1}[\mathring{R}]U[\mathring{R}]\,|j\lambda\rangle \\ &= \sum_m D^j_{m\lambda}(\mathring{R})\,|\vec{p}, jm\rangle \ . \end{aligned} \tag{2.19}$$

We shall adopt here the following normalizations for the one-particle states:

$$\begin{aligned} \langle\vec{p}\,'j'm'|\vec{p}jm\rangle &= \tilde{\delta}(\vec{p}\,' - \vec{p}\,)\delta_{jj'}\,\delta_{mm'} \\ \langle\vec{p}\,'j'\lambda'|\vec{p}j\lambda\rangle &= \tilde{\delta}(\vec{p}\,' - \vec{p}\,)\delta_{jj'}\,\delta_{\lambda\lambda'}\ , \end{aligned} \tag{2.20}$$

where $\tilde{\delta}(\vec{p}\,' - \vec{p}\,)$ is the Lorentz invariant δ-function given by

$$\tilde{\delta}(\vec{p}\,' - \vec{p}\,) = (2\pi)^3(2E)\,\delta^{(3)}(\vec{p}\,' - \vec{p}\,)\ . \tag{2.21}$$

It can be shown that, with the invariant normalization of Eq. (2.20), an arbitrary Lorentz transformation operator $U[\Lambda]$ acting on the states $|\vec{p}, jm\rangle$ or $|p, j\lambda\rangle$ is indeed a unitary operator, i.e. $U^+U = I$. With the invariant volume element as defined by

$$\tilde{\mathrm{d}}p = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3(2E)}\ , \tag{2.22}$$

the completeness relations may be written as follows:

$$\begin{aligned} \sum_{jm}\int |\vec{p}jm\rangle\ \tilde{\mathrm{d}}p\ \langle\vec{p}jm| &= I \\ \sum_{j\lambda}\int |\vec{p}j\lambda\rangle\ \tilde{\mathrm{d}}p\ \langle\vec{p}j\lambda| &= I\ , \end{aligned} \tag{2.23}$$

where I denotes the identity operator.

3 Parity and Time-Reversal Operations

Classically, the action of parity and time-reversal operations, denoted $\boldsymbol{P}$ and $\boldsymbol{T}$, may be expressed as follows:

$$\begin{aligned} \boldsymbol{P}: &\quad \vec{x} \to -\vec{x}, \quad \vec{p} \to -\vec{p}, \quad \vec{J} \to \ \ \vec{J} \\ \boldsymbol{T}: &\quad \vec{x} \to \ \ \vec{x}, \quad \vec{p} \to -\vec{p}, \quad \vec{J} \to -\vec{J} \end{aligned} \tag{3.1}$$

where $\vec{x}$, $\vec{p}$, and $\vec{J}$ stand for the coordinate, momentum, and angular momentum, respectively. It is seen from Eq. (3.1) that $\boldsymbol{P}$ and $\boldsymbol{T}$ commute with rotations, i.e.

$$[\boldsymbol{P}, R] = 0, \quad [\boldsymbol{T}, R] = 0 \ . \tag{3.2}$$

From Eq. (3.1), one sees also that the pure Lorentz transformations (in particular, boosts) act under $\boldsymbol{P}$ and $\boldsymbol{T}$ according to

$$\boldsymbol{P} L(\vec{p}\,) = L(-\vec{p}\,)\boldsymbol{P}, \qquad \boldsymbol{T} L(\vec{p}\,) = L(-\vec{p}\,)\boldsymbol{T} \ . \tag{3.3}$$

Let us now define operators acting on the physical states, representing the parity and time-reversal operations:

$$\begin{aligned} \Pi &= U[\boldsymbol{P}] \\ \mathbb{T} &= \bar{U}[\boldsymbol{T}] \ , \end{aligned} \tag{3.4}$$

where Π is a unitary operator and $\mathbb{T}$ is an anti-unitary (or anti-linear unitary) operator [1]. $\boldsymbol{T}$ is represented by an anti-unitary operator due to the fact that the time-reversal operation transforms an initial state into a final state and vice versa. Operators Π, $\mathbb{T}$, $U[R]$, and $U[L(\vec{p}\,)]$ acting on the physical states should obey the same relations as Eqs. (3.2) and (3.3):

$$\Big[\Pi, U[R]\Big] = 0 \qquad \Big[\mathbb{T}, U[R]\Big] = 0 \tag{3.5}$$

and

$$\begin{aligned} \Pi\, U[L(\vec{p}\,)] &= U[L(-\vec{p}\,)]\,\Pi \\ \mathbb{T}\, U[L(\vec{p}\,)] &= U[L(-\vec{p}\,)]\,\mathbb{T} \ . \end{aligned} \tag{3.6}$$

We are now ready to express the actions of Π and $\mathbb{T}$ on the rest states $|jm\rangle$. From the relation (3.5), it is clear that the quantum numbers j and m do not change under Π:

$$\Pi|jm\rangle = \eta|jm\rangle \ , \tag{3.7}$$

where η is the intrinsic parity of the particle represented by $|jm\rangle$. Let us write the action of $\mathbb{T}$ as follows:

$$\mathbb{T}\,|jm\rangle = \sum_k T_{km}|jk\rangle \ .$$

The relation (3.5) implies that, remembering the anti-unitarity of $\mathbb{T}$,

$$\sum_k D^j_{m'k}(R)\, T_{km} = \sum_k T_{m'k}\, D^{j\,*}_{km}(R) \ .$$

From Eqs. (A.8) and (A.9), one sees that the above relation may be satisfied, if $T_{m'm}$ is given by

$$T_{m'm} = d^j_{m'm}(\pi) = (-)^{j-m}\delta_{m',-m} \ ,$$

so that the action of $\mathbb{T}$ on the states $|jm\rangle$ may be expressed as

$$\mathbb{T}\,|jm\rangle = (-)^{j-m}|j - m\rangle \ . \tag{3.8}$$

Using the definition (2.14) and Eq. (3.6), one can show that the canonical state with momentum $\vec{p}$ transforms under Π and $\mathbb{T}$ as follows:

$$\begin{aligned} \Pi\,|\vec{p}, jm\rangle &= \eta| - \vec{p}, jm\rangle \\ \Pi\,|\phi, \theta, p, jm\rangle &= \eta|\pi + \phi, \pi - \theta, p, jm\rangle \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \mathbb{T}\,|\vec{p}, jm\rangle &= (-)^{j-m}| - \vec{p}, j - m\rangle \\ \mathbb{T}\,|\phi, \theta, p, jm\rangle &= (-)^{j-m}|\pi + \phi, \pi - \theta, p, j - m\rangle \ . \end{aligned} \tag{3.10}$$

Next, we wish to express the consequences of Π and $\mathbb{T}$ operations on the helicity states $|p, j\lambda\rangle$. The simplest way to achieve this is to use the formula (2.19), which connects the helicity and canonical states. Then, by using Eqs. (3.9), (3.10), and (A.12), one obtains easily

$$\Pi|\phi, \theta, p, j\lambda\rangle = \eta\, e^{-i\pi j}|\pi + \phi, \pi - \theta, p, j\, {-\lambda}\rangle \tag{3.11}$$

and

$$\mathbb{T}\,|\phi, \theta, p, j\lambda\rangle = e^{-i\pi\lambda}|\pi + \phi, \pi - \theta, p, j\lambda\rangle \ . \tag{3.12}$$

Now the helicity λ is an eigenvalue of $\vec{J}\cdot\hat{p}$. According to expressions (3.1), $\vec{J}\cdot\hat{p} \to -\vec{J}\cdot\hat{p}$ under $\boldsymbol{P}$ and $\vec{J}\cdot\hat{p} \to \vec{J}\cdot\hat{p}$ under $\boldsymbol{T}$. This explains why the helicity λ changes sign under Π, while it remains invariant under $\mathbb{T}$.

Finally, we wish to elaborate on the meaning of the negative momentum in the state $| - \vec{p}, jm\rangle$ in Eqs. (3.9) or (3.10). By definition,

$$| - \vec{p}, jm\rangle = U[L(-\vec{p}\,)]\,|jm\rangle \ . \tag{3.13}$$

Note the following obvious identity [see Eq. (2.7)]:

$$\begin{aligned} L(-\vec{p}\,) &= \mathring{R} L_{-z}(p) \mathring{R}^{-1} \\ &= \bar{R} L_z(p) \bar{R}^{-1} \ , \end{aligned} \tag{3.14}$$

where $L_{-z}(p)$ denotes a boost along the negative z-axis, $\mathring{R} = R(\phi, \theta, 0)$ and $\bar{R} = R(\pi + \phi, \pi - \theta, 0)$. From Eqs. (3.13) and (3.14), we obtain the result

$$| - \vec{p}, jm\rangle = |\pi + \phi, \pi - \theta, p, jm\rangle \ . \tag{3.15}$$

On the other hand, one does not have the relation like (3.15) for helicity states. Let us write

$$| - \vec{p}, j\lambda\rangle = U[\mathring{R}]U[L_{-z}(p)]\,|j\lambda\rangle \tag{3.16}$$

From Eq. (3.14), we see that

$$\overset{\circ}{R}L_{-z}(p) \neq \bar{R}L_z(p)$$

so that

$$|-\vec{p}, j\lambda\rangle \neq U[\hat{R}]U[L_z(p)]\,|j\lambda\rangle = |\pi+\phi, \pi-\theta, p, j\lambda\rangle\ . \tag{3.17}$$

The reason for this is that, while canonical states have been obtained using operators corresponding to *pure* Lorentz transformations, the helicity states are defined with operators representing a *mixture* of rotation and pure Lorentz transformation. The phase factors appearing in Eqs. (3.11) and (3.12) may be viewed as a consequence of the inequality (3.17).

4 Two-Particle States

A system consisting of two particles with arbitrary spins may be constructed in two different ways; one using the canonical basis vectors $|\vec{p}, jm\rangle$, and the other using the helicity basis vectors $|\vec{p}, j\lambda\rangle$. We shall construct in this section both the canonical and helicity states for a two-particle system having definite spin and z-component, and then derive the recoupling coefficient which connects the two bases. Afterwards, we investigate the transformation properties of the two-particle states under Π and $\mathbb{T}$, as well as the consequences of the symmetrization required when the two particles are identical.

4.1 Construction of two-particle states

Consider a system of two particles 1 and 2 with spins s_1 and s_2 and masses w_1 and w_2. In the two-particle rest frame, let $\vec{p}$ be the momentum of the particle 1, with its direction given by the spherical angles (θ, ϕ). we define the two-particle state in the canonical basis by

$$|\phi\theta m_1 m_2\rangle = a\,U[L(\vec{p}\,)]\,|s_1 m_1\rangle U[L(-\vec{p}\,)]\,|s_2 m_2\rangle\ , \tag{4.1}$$

where $|s_1 m_1\rangle$ is the rest-state of particle i and a is the normalization constant to be determined later. $L(\pm p)$ is the boost given by [see Eq. (3.14)]

$$L(\pm\vec{p}\,) = \overset{\circ}{R}(\phi,\theta,0)L_{\pm z}(p)\overset{\circ}{R}{}^{-1}(\phi,\theta,0)\ , \tag{4.2}$$

where $\overset{\circ}{R}(\phi,\theta,0)$ is again the rotation which carries the z-axis into the direction of $\vec{p}$ and $L_{\pm z}(p)$ is the boost along the $\pm z$-axis.

Owing to the rotational property (2.15) of canonical one-particle states, one may define a state of total spin s by

$$|\phi\theta s m_s\rangle = \sum_{m_1 m_2} (s_1 m_1\, s_2 m_2|s m_s)|\phi\theta m_1 m_2\rangle\ , \tag{4.3}$$

where $(s_1 m_1\, s_2 m_2|s m_s)$ is the usual Clebsch-Gordan coefficient. Using the formula (A.14), one may easily show that, if R is a rotation which takes $\Omega = (\theta, \phi)$ into $R' = R\Omega$,

$$U[R]\,|\Omega\, s m_s\rangle = \sum_{m_s'} D^s_{m_s' m_s}(R)|R'\, s m_s'\rangle\ , \tag{4.4}$$

so that the total spin s is a rotational invariant.

The state of a fixed orbital angular momentum is constructed from Eq. (4.3) in the usual way:

$$|\ell m s m_s\rangle = \int \mathrm{d}\Omega\, Y^\ell_m(\Omega)|\Omega s m_s\rangle \ , \tag{4.5}$$

where $d\Omega = d\phi\, d\cos\theta$. Let us investigate the rotational property of Eq. (4.5). Using Eq. (4.4),

$$U[R]\,|\ell m s m_s\rangle = \int \mathrm{d}\Omega\, Y^\ell_m(\Omega) D^s_{m'_s m_s}(R)|R' s m'_s\rangle \ , \tag{4.6}$$

where $R' = R'(\alpha', \beta', \gamma') = R\,\Omega,\ d\Omega = d\alpha'\, d\cos\beta'$, and, from Eqs. (A.13) and (A.3),

$$\begin{aligned} Y^\ell_m(\Omega) &= \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell *}_{m0}(R^{-1}R') \\ &= \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m'} D^{\ell *}_{mm'}(R^{-1})\, D^{\ell *}_{m'0}(R') \\ &= \sum_{m'} D^\ell_{m'm}(R) Y^\ell_{m'}(\beta', \alpha') \ , \end{aligned} \tag{4.7}$$

one obtains the result

$$U[R]\,|\ell m s m_s\rangle = \sum_{m' m'_s} D^\ell_{m'm}(R) D^s_{m'_s m_s}(R)|\ell m' s m'_s\rangle \ . \tag{4.8}$$

This shows that the states $|\ell m s m_s\rangle$ transform under rotation as a product of two "rest states" $|\ell m\rangle$ and $|s m_s\rangle$.

Now, it is easy to construct a state of total angular momentum J:

$$\begin{aligned} |JM\ell s\rangle &= \sum_{m m_s} (\ell m s m_s|JM)|\ell m s m_s\rangle \\ &= \sum_{\substack{m m_s \\ m_1 m_2}} (\ell m s m_s|JM)(s_1 m_1\, s_2 m_2|s m_s) \times \\ &\qquad \times \int \mathrm{d}\Omega\, Y^\ell_m(\Omega)|\Omega m_1 m_2\rangle \ . \end{aligned} \tag{4.9}$$

From Eqs. (4.8) and (A.14), one sees immediately

$$U[R]\,|JM\ell s\rangle = \sum_{M'} D^J_{M'M}(R)|JM'\ell s\rangle \ . \tag{4.10}$$

Note that, as expected, ℓ and s are rotational invariants: Eq. (4.9) is the equivalent of the non-relativistic L-S coupling.

Next, we turn to the problem of constructing two-particle states from the helicity basis vectors $|\vec{p}, j\lambda\rangle$. In analogy to Eq. (4.1), we write

$$\begin{aligned} |\phi\theta\lambda_1\lambda_2\rangle &= a\, U[\mathring{R}]\Big\{U[L_z(p)]\,|s_1\lambda_1\rangle U[L_{-z}(p)]\,|s_2\ -\lambda_2\rangle\Big\} \\ &\equiv U[\mathring{R}(\phi, \theta, 0)]\,|00\lambda_1\lambda_2\rangle \ , \end{aligned} \tag{4.11}$$

where $|s_i\lambda_i\rangle$ is the rest state of particle i and a the normalization constant of Eq. (4.1). We have constructed the helicity state for the particle 2 in such a way that its helicity quantum number is $+\lambda_2$.

States of definite angular momentum J may be constructed from Eq. (4.11) as follows:

$$|JM\lambda_1\lambda_2\rangle = \frac{N_J}{2\pi}\int \mathrm{d}R\, D^{J\,*}_{M\mu}(R)U[R]\,|00\lambda_1\lambda_2\rangle \ , \tag{4.12}$$

where N_J is a normalization constant to be determined later. Let us apply an arbitrary rotation R' on the state (4.12):

$$U[R'\,]\,|JM\lambda_1\lambda_2\rangle = \frac{N_J}{2\pi}\int \mathrm{d}R\, D^{J\,*}_{M\mu}(R)U[R''\,]\,|00\lambda_1\lambda_2\rangle \ ,$$

where $R'' = R'R$. But, by using Eq. (A.3) and the unitarity of the D-functions,

$$\begin{aligned}
D^{J\,*}_{M\mu}(R) &= D^{J\,*}_{M\mu}(R'^{\,-1}R''\,) \\
&= \sum_{M'} D^{J\,*}_{MM'}(R'^{\,-1})D^{J\,*}_{M'\mu}(R''\,) \\
&= \sum_{M'} D^{J}_{M'M}(R')\,D^{J\,*}_{M'\mu}(R''\,)\ .
\end{aligned}$$

Using this relation, as well as the fact that $dR = dR''$, one obtains the result

$$U[R'\,]\,|JM\lambda_1\lambda_2\rangle = \sum_{M'} D^{J}_{M'M}(R'\,)|JM'\lambda_1\lambda_2\rangle \ , \tag{4.13}$$

so that states (4.12) are indeed states of a definite angular momentum J. Note that, as expected, λ_1 and λ_2 are rotational invariants.

Now, let us specify the rotation R appearing in Eq. (4.12) by writing $R = R(\phi,\theta,\gamma)$. Then,

$$\begin{aligned}
&U[R(\phi,\theta,\gamma)]\,|00\lambda_1\lambda_2\rangle \\
=\ & U[R(\phi,\theta,0)]U[R(0,0,\gamma)]\,|00\lambda_1\lambda_2\rangle \\
=\ & e^{-i(\lambda_1-\lambda_2)\,\gamma}\,U[R(\phi,\theta,0)]\,|00\lambda_1\lambda_2\rangle\ .
\end{aligned} \tag{4.14}$$

The last relation follows because of the commutation relation

$$[R(0,0,\gamma), L_{\pm z}(p)] = 0\ . \tag{4.15}$$

Substituting Eq. (4.14) into Eq. (4.12), and integrating over $d\gamma$, one obtains

$$|JM\lambda_1\lambda_2\rangle = N_J\int \mathrm{d}\Omega\, D^{J\,*}_{M\lambda}(\phi,\theta,0)|\phi\theta\lambda_1\lambda_2\rangle \ , \tag{4.16}$$

where $\lambda = \lambda_1 - \lambda_2$.

4.2 Normalization

We shall now specify the normalizations we adopt for states (4.1) and (4.11). The most convenient choices are

$$\langle\Omega' m_1' m_2'|\Omega m_1 m_2\rangle = \delta^{(2)}(\Omega' - \Omega)\delta_{m_1 m_1'}\delta_{m_2 m_2'} \tag{4.17}$$

and

$$\langle\Omega'\lambda_1'\lambda_2'|\Omega\lambda_1\lambda_2\rangle = \delta^{(2)}(\Omega' - \Omega)\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} \ . \tag{4.18}$$

With the single particle normalizations as defined in Eqs. (2.20), one may show (see Appendix C) that

$$a = \frac{1}{4\pi}\sqrt{\frac{p}{w}} \ , \tag{4.19}$$

where p is the relative momentum and w is the effective mass of the two-particle system. The normalization (4.17) implies that the states $|JM\ell s\rangle$ given in Eq. (4.9) obey the following normalizations:

$$\langle J'M'\ell' s'|JM\ell s\rangle = \delta_{JJ'}\delta_{MM'}\delta_{\ell\ell'}\delta_{ss'} \ . \tag{4.20}$$

From Eqs. (4.18) and (A.4), the state $|JM\lambda_1\lambda_2\rangle$ of formula (4.16) is seen to be normalized according to

$$\langle J'M'\lambda_1'\lambda_2'|JM\lambda_1\lambda_2\rangle = \delta_{JJ'}\delta_{MM'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} \ , \tag{4.21}$$

if the constant N_J is set equal to

$$N_J = \sqrt{\frac{2J+1}{4\pi}} \ . \tag{4.22}$$

The completeness relations may now be written

$$\sum_{\substack{JM\\ \ell s}} |JM\ell s\rangle\langle JM\ell s| = I \tag{4.23}$$

and

$$\sum_{\substack{JM\\ \lambda_1\lambda_2}} |JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2| = I \ . \tag{4.24}$$

From Eqs. (4.16) and (4.18), we obtain the relation

$$\langle\Omega\lambda_1'\lambda_2'|JM\lambda_1\lambda_2\rangle = N_J D^{J\,*}_{M\lambda}(\phi,\theta,0)\,\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} \ . \tag{4.25}$$

4.3 Connection between canonical and helicity states

We start from formula (4.11):

$$\begin{aligned}
|\phi\theta\lambda_1\lambda_2\rangle &= a\,U[\mathring{R}]\Big\{U[L_z(p)]\,|s_1\lambda_1\rangle U[L_{-z}(p)]\,|s_2\,{-\lambda_2}\rangle\Big\} \\
&= a\,U[L(\vec{p}\,)]U[\mathring{R}]\,|s_1\lambda_1\rangle U[L(-\vec{p}\,)]U[\mathring{R}]\,|s_2\,{-\lambda_2}\rangle \\
&= \sum_{m_1 m_2} D^{s_1}_{m_1\lambda_1}(\phi,\theta,0)D^{s_2}_{m_2-\lambda_2}(\phi,\theta,0)|\phi\theta m_1 m_2\rangle \ ,
\end{aligned} \tag{4.26}$$

where we have used the formulae (2.12), (1.5), and (4.1). Then from Eq. (4.16),

$$|JM\lambda_1\lambda_2\rangle = N_J \sum_{m_1m_2} \int d\Omega\, D^{J\,*}_{M\lambda}(\phi,\theta,0) D^{s_1}_{m_1\lambda_1}(\phi,\theta,0) D^{s_2}_{m_2-\lambda_2}(\phi,\theta,0) |\phi\theta m_1m_2\rangle \ . \tag{4.27}$$

The product of three D-functions appearing in Eq. (4.27) may be reduced as follows. From Eq. (A.14),

$$D^{s_1}_{m_1\lambda_1} D^{s_2}_{m_2-\lambda_2} = \sum_{sm_s} (s_1m_1s_2m_2|sm_s)(s_1\lambda_1s_2-\lambda_2|s\lambda) D^s_{m_s\lambda} \tag{4.28}$$

and, from Eq. (A.15),

$$D^{J\,*}_{M\lambda} D^s_{m_s\lambda} = \sum_{\ell m} \sqrt{\frac{4\pi}{2\ell+1}} \left(\frac{2\ell+1}{2J+1}\right) (\ell m s m_s|JM)(\ell 0 s\lambda|J\lambda) Y^\ell_m \ . \tag{4.29}$$

Substituting these into Eq. (4.27) and comparing the result with Eq. (4.9), we obtain finally

$$|JM\lambda_1\lambda_2\rangle = \sum_{\ell s} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0 s\lambda|J\lambda)(s_1\lambda_1 s_2 -\lambda_2|s\lambda)|JM\ell s\rangle \ , \tag{4.30}$$

so that the recoupling coefficient between canonical and helicity states is given by

$$\langle J'M'\ell s|JM\lambda_1\lambda_2\rangle = \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0 s\lambda|J\lambda)(s_1\lambda_1 s_2 -\lambda_2|s\lambda)\delta_{JJ'}\delta_{MM'} \ . \tag{4.31}$$

The relation (4.30) may be inverted to give

$$\begin{aligned} |JM\ell s\rangle &= \sum_{\lambda_1\lambda_2} |JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2|JM\ell s\rangle \\ &= \sum_{\lambda_1\lambda_2} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0 s\lambda|J\lambda)(s_1\lambda_1 s_2 -\lambda_2|s\lambda)|JM\lambda_1\lambda_2\rangle \ . \end{aligned} \tag{4.32}$$

4.4 Symmetry relations

The canonical states $|JM\ell s\rangle$ transform in a particularly simple manner under symmetry operations (e.g. parity and time-reversal), and the derivation is also much simpler than for helicity states. For this reason, we shall first investigate the consequences of symmetry operations on the canonical states, and then obtain the corresponding relations for the states $|JM\lambda_1\lambda_2\rangle$ by using the relation (4.30).

We shall first start with the parity operation. Using the formula (3.9), we find easily

$$\Pi|\phi\theta m_1m_2\rangle = \eta_1\eta_2|\pi+\phi,\pi-\theta,m_1m_2\rangle \ , \tag{4.33}$$

where $\eta_1(\eta_2)$ is the intrinsic parity of particle 1(2). From the defining equation (4.9), we then obtain immediately

$$\Pi|JM\ell s\rangle = \eta_1\eta_2(-)^\ell|JM\ell s\rangle \ , \tag{4.34}$$

so that the "ℓ-s coupled" states are in an eigenstate of Π with the eigenvalue $\eta_1\eta_2(-)^\ell$, a well known result. Using the formula (4.30) and the symmetry relations of Clebsch-Gordan coefficients, one finds for the helicity states

$$\Pi\,|JM\lambda_1\lambda_2\rangle = \eta_1\eta_2(-)^{J-s_1-s_2}|JM\,{-\lambda_1}\,{-\lambda_2}\rangle\ . \tag{4.35}$$

Again, the helicities reverse sign, as was the case for the single-particle states [see Eq. (3.11)].

Consequences of the time-reversal operation may be explored in a similar fashion. Using Eq. (3.10), one finds immediately

$$\mathbb{T}\,|\phi\theta m_1 m_2\rangle = (-)^{s_1-m_1}(-)^{s_2-m_2}|\pi+\phi, \pi-\theta, -m_1-m_2\rangle\ . \tag{4.36}$$

Then, from Eq. (4.9),

$$\mathbb{T}\,|JM\ell s\rangle = (-)^{J-M}|J-M\ell s\rangle \tag{4.37}$$

and, from Eq. (4.30),

$$\mathbb{T}\,|JM\lambda_1\lambda_2\rangle = (-)^{J-M}|J-M\lambda_1\lambda_2\rangle\ . \tag{4.38}$$

Now, we investigate the effects of symmetrization required when the particles 1 and 2 are identical. Regardless of whether the particles are bosons or fermions, the symmetrized state may be written, for the canonical states

$$|JM\ell s\rangle_s = a_s[1+(-)^{2s_1}\,\mathbb{P}_{12}]\,|JM\ell s\rangle\ , \tag{4.39}$$

where $\mathbb{P}_{12}$ is the particle-exchange operator and a_s is the normalization constant. Again, using the defining equation (4.9), one obtains

$$\mathbb{P}_{12}|JM\ell s\rangle = (-)^{\ell+s-2s_1}|JM\ell s\rangle$$

or

$$|JM\ell s\rangle_s = a_s[1+(-)^{\ell+s}]\,|JM\ell s\rangle\ , \tag{4.40}$$

so that $\ell+s=$ even for a system of identical particles in an eigenstate of orbital angular momentum ℓ and total spin s and $a_s=1/2$. Now, the symmetrized helicity state may be written

$$|JM\lambda_1\lambda_2\rangle_s = b_s(\lambda_1\lambda_2)[1+(-)^{2s_1}\,\mathbb{P}_{12}]\,|JM\lambda_1\lambda_2\rangle\ , \tag{4.41}$$

where $b_s(\lambda_1\lambda_2)$ is the normalization constant. From Eqs. (4.30) and (4.40), one finds

$$|JM\lambda_1\lambda_2\rangle_s = b_s(\lambda_1\lambda_2)\left\{|JM\lambda_1\lambda_2\rangle + (-)^J|JM\lambda_2\lambda_1\rangle\right\}\ , \tag{4.42}$$

where $b_s(\lambda_1\lambda_2)=1/\sqrt{2}$ for $\lambda_1\neq\lambda_2$ and $b_s(\lambda_1\lambda_2)=1/2$ for $\lambda_1=\lambda_2$. Note that, for a system of identical particles, the symmetrized states in both canonical and helicity bases have the same forms, regardless of whether the particles involved are fermions or bosons.

5 Applications

We are now ready to apply results of the previous section to a few physical problems of practical importance. As a first application, we shall write down the invariant transition amplitude for two-body reactions and derive the partial-wave expansion formula. We do this in the helicity basis, following the derivation given in the "classic" paper by Jacob and Wick [14]. Our main purpose in this exercise is to show how the particular normalization (2.20) of single-particle states influences the precise definition of the invariant amplitudes and the corresponding cross-section formula (see Appendix B).

Next, we shall discuss the general two-body decays of resonances and give the symmetry relations satisfied by the decay amplitude, as well as the coupling formula which connects the helicity decay amplitude to the partial-wave amplitudes. Finally, we take up the discussion of the spin density matrices, introduce the multipole parameters, and then expand the angular distribution for two-body decays in terms of the multipole parameters.

5.1 S-matrix for two-body reactions

Let us denote a two-body reaction by

$$a + b \to c + d \tag{5.1}$$

with $\vec{p}_a$, s_a, λ_a, and η_a standing for the momentum, spin, helicity, and the intrinsic parity of the particle a, etc. Let w_0 denote the centre-of-mass (c.m.) energy and let $\vec{p}_i(\vec{p}_f)$ be the c.m. momentum of the particle $a(c)$. The invariant S-matrix element for the reaction (5.1) may be written, in the over-all c.m. system,

$$\begin{aligned} \langle \vec{p}_c\lambda_c; \vec{p}_d\lambda_d|S|\vec{p}_a\lambda_a; \vec{p}_b\lambda_b\rangle &= \langle \vec{p}_f\lambda_c; -\vec{p}_f\lambda_d|S|\vec{p}_i\lambda_a; -\vec{p}_i\lambda_b\rangle \\ &= (4\pi)^2 \frac{w_0}{\sqrt{p_f p_i}} \langle \Omega_0\lambda_c\lambda_d|S|00\lambda_a\lambda_b\rangle \ , \end{aligned} \tag{5.2}$$

where we have used Eq. (4.11) with the normalization constant as given in Eq. (4.19), and we have fixed the direction $\vec{p}_i$ at the spherical angles $(0,0)$ and $\vec{p}_f$ at $\Omega_0 = (\theta_0, \phi_0)$. Because of the invariant normalization (2.20) of the one-particle states, the absolute square of the amplitude (5.2) summed over the helicities λ_a, λ_b, etc., is a Lorentz invariant quantity. It is in this sense that formula (5.2) is referred to as the "invariant S matrix". Due to the energy-momentum conservation, one may write

$$\langle \Omega_0\lambda_c\lambda_d|S|00\lambda_a\lambda_b\rangle = (2\pi)^4\delta^{(4)}(p_c + p_d - p_a - p_b)\ \langle \Omega_0\lambda_c\lambda_d|S(w_0)|00\lambda_a\lambda_b\rangle \ . \tag{5.3}$$

If we define the T operator via $S = 1 + iT$, it is clear that we may write down the T-matrix in the same way as in formulae (5.2) and (5.3), simply replacing S by T. Now, the *invariant* transition amplitude $\mathcal{M}_{fi}$ is defined from the T matrix by

$$(2\pi)^4\delta^{(4)}(p_c + p_d - p_a - p_b)\ \mathcal{M}_{fi} = \langle p_c\lambda_c; p_d\lambda_d|T|p_a\lambda_a; p_b\lambda_b\rangle \tag{5.4}$$

or

$$\mathcal{M}_{fi} = (4\pi)^2 \frac{w_0}{\sqrt{p_f p_i}} \langle \Omega_0\lambda_c\lambda_d|T(w_0)|00\lambda_a\lambda_b\rangle \ . \tag{5.5}$$

The differential cross-section for fixed helicities is related to the transition amplitude by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_0} = \frac{p_f}{p_i}\left|\frac{\mathcal{M}_{fi}}{8\pi w_0}\right|^2 , \tag{5.6}$$

which has been obtained using Eqs. (B.2), (B.3), and (B.6).

Let us now expand the transition amplitude in terms of the partial-wave amplitudes:

$$\begin{aligned}\langle \Omega_0\lambda_c\lambda_d|T(w_0)|00\lambda_a\lambda_b\rangle &= \sum_{JM}\langle \Omega_0\lambda_c\lambda_d|JM\lambda_c\lambda_d\rangle\langle JM\lambda_c\lambda_d|T(w_0)|JM\lambda_a\lambda_b\rangle \\ &\qquad \times\langle JM\lambda_a\lambda_b|00\lambda_a\lambda_b\rangle \\ &= \frac{1}{4\pi}\sum_J(2J+1)\langle\lambda_c\lambda_d|T^J(w_0)|\lambda_a\lambda_b\rangle D^{J\,*}_{\lambda\lambda'}(\phi_0,\theta_0,0) ,\end{aligned} \tag{5.7}$$

where $\lambda = \lambda_a - \lambda_b$ and $\lambda' = \lambda_c - \lambda_d$.

If we define the "scattering amplitude" $f(\Omega_0)$ via

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_0} = |f(\Omega_0)|^2 \tag{5.8}$$

we obtain

$$f(\Omega_0) = \frac{(p_f/p_i)^{\frac{1}{2}}}{8\pi w_0}\mathcal{M}_{fi} . \tag{5.9}$$

This formula then relates to the "non-relativistic" scattering amplitude $f(\Omega_0)$ to the Lorentz invariant transition amplitude $\mathcal{M}_{fi}$. From Eqs. (5.5) and (5.7), one sees immediately that

$$f(\Omega_0) = \frac{1}{p_i}\sum_J\left(J+\frac{1}{2}\right)\langle\lambda_c\lambda_d|T^J(w_0)|\lambda_a\lambda_b\rangle D^{J\,*}_{\lambda\lambda'}(\phi_0,\theta_0,0) . \tag{5.10}$$

The partial-wave T-matrix appearing in Eq. (5.10) is related to the partial-wave S-matrix by

$$\langle\lambda_c\lambda_d|S^J(w_0)|\lambda_a\lambda_b\rangle = \delta_{fi}\delta_{\lambda_c\lambda_a}\delta_{\lambda_d\lambda_b} + i\langle\lambda_c\lambda_d|T^J(w)|\lambda_a\lambda_b\rangle , \tag{5.11}$$

where $\delta_{fi} = 1$ for elastic scattering and zero, otherwise.

If parity is conserved in the process (5.1), it follows from Eq. (4.35) that the partial-wave amplitude given by Eq. (5.11) should satisfy the following symmetry relation:

$$\langle -\lambda_c - \lambda_d|S^J(w_0)| - \lambda_a - \lambda_b\rangle = \eta\,\langle\lambda_c\lambda_d|S^J(w_0)|\lambda_a\lambda_b\rangle , \tag{5.12}$$

where

$$\eta = \frac{\eta_c\eta_d}{\eta_a\eta_b}(-)^{s_c+s_d-s_a-s_b} .$$

Next, we examine the consequences of time-reversal invariance. Let us denote by $|i\rangle$ and $|f\rangle$ the initial and final system in a scattering process. Then, the time-reversed process takes the initial state $|\boldsymbol{T}f\rangle$ into the final state $|\boldsymbol{T}i\rangle$, so that time-reversal invariance implies the following relation for the S-matrix:

$$\langle f|S|i\rangle = \langle\boldsymbol{T}i|S|\boldsymbol{T}f\rangle . \tag{5.13}$$

Using Eq. (4.38), one finds immediately

$$\langle\lambda_c\lambda_d|S^J(w_0)|\lambda_a\lambda_b\rangle = \langle\lambda_a\lambda_b|S^J(w_0)|\lambda_c\lambda_d\rangle , \tag{5.14}$$

where the right-hand side refers to the process $c + d \to a + b$.

5.2 Two-body decays

Let us consider a resonance of spin-parity J^η and mass w (to be called the resonance J), decaying into a two-particle system with particles 1 and 2:

$$J \to 1 + 2 \,, \tag{5.15}$$

and let $s_1(s_2)$ and $\eta_1(\eta_2)$ denote the spin and intrinsic parity of the particle 1(2). In the rest frame of the resonance $J(J\mathrm{RF})$, let $\vec{p}$ be the momentum of the particle 1 with the spherical angles given by $\Omega = (\theta, \phi)$. Then, the amplitude A describing the decay of spin J with the z-component M into two particles with helicities λ_1 and λ_2 may be written

$$\begin{aligned} A &= \langle \vec{p}\lambda_1; -\vec{p}\lambda_2|\mathcal{M}|JM\rangle \\ &= 4\pi \left(\frac{w}{p}\right)^{\frac{1}{2}} \langle \phi\theta\lambda_1\lambda_2|JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle \\ &= N_J F^J_{\lambda_1\lambda_2} D^{J\,*}_{M\lambda}(\phi,\theta,0), \qquad \lambda = \lambda_1 - \lambda_2 \,, \end{aligned} \tag{5.16}$$

where one has used the formulae (4.19), (4.24), and (4.25). The "helicity decay amplitude" F is given by

$$F^J_{\lambda_1\lambda_2} = 4\pi \left(\frac{w}{p}\right)^{\frac{1}{2}} \langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle \,. \tag{5.17}$$

Since $\mathcal{M}$ is a rotational invariant, the helicity amplitude F can depend only on the rotationally invariant quantities, namely, J, λ_1, and λ_2.

It is easy to expand the helicity decay amplitude F in terms of the partial-wave amplitudes. Using the recoupling coefficient (4.31), we may write

$$\begin{aligned} \langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle &= \sum_{\ell s} \langle JM\lambda_1\lambda_2|JM\ell s\rangle\langle JM\ell s|\mathcal{M}|JM\rangle \\ &= \sum_{\ell s} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0 s\lambda|J\lambda)(s_1\lambda_1 s_2 \, {-\lambda_2}|s\lambda)\langle JM\ell s|\mathcal{M}|JM\rangle \end{aligned}$$

so that F may be expressed

$$F^J_{\lambda_1\lambda_2} = \sum_{\ell s} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} a^J_{\ell s} \, (\ell 0 s\lambda|J\lambda) \, (s_1\lambda_2 s_2 \, {-\lambda_2}|s\lambda) \,, \tag{5.18}$$

where the partial-wave amplitude $a^J_{\ell s}$ is defined by

$$a^J_{\ell s} = 4\pi \left(\frac{w}{p}\right)^{\frac{1}{2}} \langle JM\ell s|\mathcal{M}|JM\rangle \,. \tag{5.19}$$

The normalizations have a simple relationship

$$\sum_{\lambda_1\,\lambda_2} |F^J_{\lambda_1\lambda_2}|^2 = \sum_{\ell\, s} |a^J_{\ell s}|^2 \tag{5.19a}$$

If parity is conserved in the decay, we have, from Eq. (4.35),

$$F^J_{\lambda_1\lambda_2} = \eta\eta_1\eta_2(-)^{J-s_1-s_2}F^J_{-\lambda_1-\lambda_2} \; , \tag{5.20}$$

where η_1 and η_2 are the intrinsic parities of the particles 1 and 2. If the particles 1 and 2 are identical, we have to replace the state $|JM\lambda_1\lambda_2\rangle$ in Eq. (5.17) by the symmetrized state of Eq. (4.42), so that we obtain the following symmetry relation:

$$F^J_{\lambda_1\lambda_2} = (-)^J F^J_{\lambda_2\lambda_1} \; . \tag{5.21}$$

It is possible to obtain a further symmetry relation on F by considering the time-reversal operations. For the purpose, let us consider the elastic scattering of particles 1 and 2 in the angular momentum state $|JM\lambda_1\lambda_2\rangle$, i.e.

$$\langle JM\lambda_1'\lambda_2'|T(w)|JM\lambda_1\lambda_2\rangle \equiv \langle\lambda_1'\lambda_2'|T^J(w)|\lambda_1\lambda_2\rangle \; , \tag{5.22}$$

where w is the c.m. energy and coincides with the effective mass of the resonance J. Now, we make the assumption that the J^{th} partial wave for the elastic scattering of particles 1 and 2 is completely dominated by the resonance at the c.m. energy w (see Fig. 1.2).

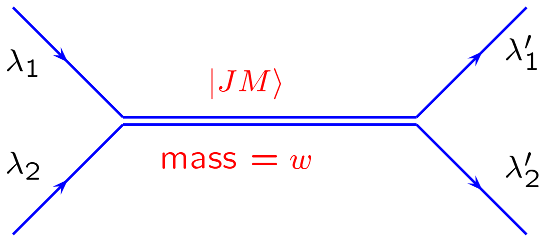

Figure 1.2: Elastic scattering of particles 1 and 2, mediated by a resonance J in the s-channel.

Then, we may write

$$T(w) \sim \sum_M \mathcal{M}|JM\rangle D(w)\langle JM|\mathcal{M}^\dagger \; ,$$

where $D(w)$ is the Breit-Wigner function for the resonance and $\mathcal{M}$ is the "decay operator" of Eq. (5.17). Substituting this into Eq. (5.22), we obtain

$$\langle\lambda'\lambda_2'|T^J(w)|\lambda_1\lambda_2\rangle \sim D(w)F^J_{\lambda_1'\lambda_2'}F^{J\,*}_{\lambda_1\lambda_2} \; ,$$

so that time-reversal invariance for elastic scattering implies, from Eqs. (5.11) and (5.14),

$$F^J_{\lambda_1'\lambda_2'}F^{J\,*}_{\lambda_1\lambda_2} = F^J_{\lambda_1\lambda_2}F^{J\,*}_{\lambda_1'\lambda_2'} \; . \tag{5.23}$$

This means that the phase of the complex amplitude F does not depend on the helicities λ_1 and λ_2. Therefore, we can consider F a real quantity without loss of generality:

$$F^J_{\lambda_1\lambda_2} = \text{real} \; . \tag{5.24}$$

We emphasize that this result follows only from the assumption that the J^{th} partial wave is dominated by the resonance J at the energy w. This condition is fulfilled, for example, in the P-wave amplitudes of the $\pi^+\pi^-$ or $p\pi^+$ elastic scattering at the c.m. energies corresponding to ρ^0 and $\Delta(1232)$ masses, where it is known that these resonances saturate the unitarity limit. It is clear, however, that this condition may not be satisfied for all resonances. In this sense, the relations (5.23) or (5.24) may be considered only an "approximate" symmetry. We will show later in the discussion of the sequential decay modes that the symmetry (5.23) can actually be tested experimentally.

Before we proceed to a discussion of the angular distribution resulting from the decay of a resonance of spin J, it is necessary to construct the corresponding spin density matrix, which carries the information on how the resonance has actually been produced.

5.3 Density matrix and angular distribution

Let us consider the production and decay of a resonance J given by

$$a + b \to c + J, \qquad J \to 1 + 2\ . \tag{5.25}$$

We shall use for this process the same notations, wherever possible, as those of Sections 5.1 and 5.2. Note, however, that the helicity corresponding to the resonance J is denoted by Λ, and w is the effective mass of the particles 1 and 2. The over-all transition amplitude $\mathcal{M}_{fi}$ may be written, combining Eqs. (5.5) and (5.16),

$$\mathcal{M}_{fi} \sim \sum_{\Lambda} \langle \vec{p}\lambda_1\lambda_2|\mathcal{M}|J\Lambda\rangle\langle\vec{p}_f\lambda_c\Lambda|T(w_0)|\vec{p}_i\lambda_a\lambda_b\rangle\ . \tag{5.26}$$

The differential cross-section in the JRF decay angles $\Omega = (\theta,\phi)$ may be expressed, after summing over all other variables except Ω,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim \int \mathrm{d}\Omega_0\,\mathrm{d}w K(w) \sum |\mathcal{M}_{fi}|^2\ , \tag{5.27}$$

where $K(w)$ is a factor which includes all the quantities dependent on w, such as the phase space factor [see Eq. (B.8)] and the square of the Breit-Wigner function $D(w)$ of the resonance J.

Next, we introduce the spin density matrix corresponding to the resonance J:

$$\rho^J_{\Lambda\Lambda'} \sim \int \mathrm{d}\Omega_0 \sum \langle\vec{p}_f\lambda_c\Lambda|T(w_0)|\vec{p}_i\lambda_a\lambda_b\rangle\ \langle\vec{p}_f\lambda_c\Lambda|T(w_0)|\vec{p}_i\lambda_a\lambda_b\rangle^*\ , \tag{5.28}$$

where the summation sign denotes the sum over λ_a, λ_b, and λ_c. Then, from expressions (5.26) and (5.27),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim \int \mathrm{d}w\, K(w) \sum_{\substack{\Lambda\Lambda'\\ \lambda_1\lambda_2}} \langle\vec{p}\lambda_1\lambda_2|\mathcal{M}|J\Lambda\rangle\rho^J_{\Lambda\Lambda'}\langle J\Lambda'|\mathcal{M}^\dagger|\vec{p}\lambda_1\lambda_2\rangle \tag{5.29}$$

One sometimes defines the density matrix by

$$\rho^J = \sum_{\Lambda\Lambda'} |J\Lambda\rangle \rho^J_{\Lambda\Lambda'} \langle J\Lambda'| \ . \tag{5.30}$$

Then,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim \int \mathrm{d}w\, K(w) \sum_{\lambda_1\lambda_2} \langle \vec{p}\lambda_1\lambda_2 | \mathcal{M}\rho^J \mathcal{M}^\dagger | \vec{p}\lambda_1\lambda_2 \rangle \ .$$

At this point, we introduce a simplifying assumption that $rho^J_{\Lambda\Lambda'}$, is independent of w over the width of the resonance J. This assumption makes the resulting formalism much simpler. It can be shown that a more general formalism without this simplifying assumption leads to identical results in most cases [see Chung[15]]. We shall come back to this point later, when we discuss the sequential decay modes.

Now, we can absorb the integration over dw into the decay amplitude F, and define

$$g^J_{\lambda_1\lambda_2} \sim \int \mathrm{d}w\, K(w) |F^J_{\lambda_1\lambda_2}|^2 \ . \tag{5.31}$$

We emphasize that F is in general a complicated function of w. If the partial-wave amplitude is proportional to p^ℓ, we see from Eq. (5.18) that F is a function of w in a way that makes it impossible to "split off" a helicity-independent function of w from F.

Combining expressions (5.16), (5.29), and (5.31), we obtain the explicit expression for the differential cross-section ($\lambda = \lambda_1 - \lambda_2$)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim N_J^2 \sum_{\lambda_1\lambda_2}^{\Lambda\Lambda'} \rho^J_{\Lambda\Lambda'} D^{J\,*}_{\Lambda\lambda}(\phi,\theta,0) g^J_{\lambda_1\lambda_2} \ . \tag{5.32}$$

Let us denote by $I(\Omega)$ the normalized angular distribution, i.e.

$$\int \mathrm{d}\Omega\, I(\Omega) = 1 \ . \tag{5.33}$$

Then, we may write

$$I(\Omega) = \left(\frac{2J+1}{4\pi}\right) \sum_{\substack{\Lambda\Lambda' \\ \lambda_1\lambda_2}} \rho^J_{\Lambda\Lambda'} D^{J\,*}_{\Lambda\lambda}(\phi,\theta,0) D^J_{\Lambda'\lambda}(\phi,\theta,0) g^J_{\lambda_1\lambda_2} \ . \tag{5.34}$$

$I(\Omega)$ is a normalized distribution, if we impose:

$$\sum_{\Lambda} \rho^J_{\Lambda\Lambda} = 1 \tag{5.35}$$

and

$$\sum_{\lambda_1\lambda_2} g^J_{\lambda_1\lambda_2} = 1 \ . \tag{5.36}$$

Note that $I(\Omega)$ is real, as it should be. This can be shown easily by using the fact that the density matrix is Hermitian by definition [see Eq. (5.28)].

Next, we turn to a discussion of the symmetry relations for $\rho^H_{\Lambda\Lambda'}$ coming from parity conservation in the production process. We fix the production coordinate system such that the reaction $a+b \to c+J$ takes place in the x-z plane. Consequences of parity conservation may now be investigated using the reflection operator through the y-axis:

$$\Pi_y = \Pi e^{-i\pi J_y} \ . \tag{5.37}$$

It is clear that this operator commutes with any operator representing a rotation around the y-axis:

$$\Big[\Pi_y, U[R_y]\Big] = 0 \ . \tag{5.38}$$

In addition, it will commute with any operator representing a boost in the x-z plane:

$$\Big[\Pi_y, U[L(\vec{K}\ )]\Big] = 0 \ , \tag{5.39}$$

where the momentum $\vec{K}$ lies in the x-z plane.

It is now easy to see that the Π_y acting on the T-matrix in Eq. (5.28) will leave the momenta $\vec{p}_f$ and $\vec{p}_i$ unchanged and act directly on the rest states:

$$\Pi_y|s_i\lambda_i\rangle = \eta_i(-)^{s_i-\lambda_i}|s_i\ {-\lambda_i}\rangle \ , \tag{5.40}$$

where the index i stands for the particles a, b, c, or J. substituting Eq. (5.40) into Eq. (5.28), one obtains the result

$$\rho^J_{\Lambda\Lambda'} = (-)^{\Lambda-\Lambda'}\rho^J_{-\Lambda-\Lambda'} \ . \tag{5.41}$$

Note that, owing to the relation (5.39), the density matrix defined in the canonical basis instead of the helicity basis will satisfy the same symmetry relation as in Eq. (5.41). In fact, expression (5.38) implies that the symmetry given in Eq. (5.41) is true as long as the quantization axis remains in the production plane.

We shall derive another symmetry relation applicable to the density matrix defined in the canonical basis. The canonical density matrix is, of course, obtained by replacing λ_i in Eq. (5.28) by m_i, the z-component of the spin s_i in the canonical description. Suppose now that the production process $a+b \to c+J$ takes place in the x-y plane. It is convenient, in this case, to define a reflection operator through the z-axis:

$$\Pi_z = \Pi e^{-i\pi Jz} \ . \tag{5.42}$$

Note that, in analogy to expression (5.39),

$$\Big[\Pi_z, U[L(\vec{q}\ )]\Big] = 0 \ , \tag{5.43}$$

where the momentum $\vec{q}$ lies in the x-y plane.

Therefore, the Π_z acting on the T-matrix in Eq. (5.28) leaves the momenta $\vec{p}_f$ and $\vec{p}_i$ in peace, while the Π_z acting on the rest states $|s_im_i\rangle$ brings out a phase factor

$$\Pi_z|s_im_i\rangle = \eta_i e^{-im_i\pi}|s_im_i\rangle \ . \tag{5.44}$$

Now, the resulting symmetry relation can be written down easily:

$$\rho^J_{mm'} = (-)^{m-m'} \rho^J_{mm'} \ , \tag{5.45}$$

where m is the z-component of spin J in the canonical basis. The relation (5.45) implies that $\rho^J_{mm'} = 0$ if $m - m'$ is odd; this symmetry is known as Capps' checker-board theorem [10].

We are now ready to examine the implications of parity conservation in the angular distribution given by Eq. (5.34). For the purpose, we choose the Jackson frame for the resonance J, i.e. the z-axis along the direction $\vec{p}_b$ and the y-axis along the production normal in the JRF. Applying the symmetry (5.41) and the formula (A.12) for the D-functions to the angular distribution (5.34), we obtain

$$I(\theta, \phi) = I(\pi - \theta, \pi - \phi) \ . \tag{5.46}$$

This is then the general symmetry relation applicable if the quantization axis is in the production plane, regardless of whether the parity is conserved in the *decay* process.

Integrating over the angle ϕm the angular distribution is seen to satisfy: $I_1(\theta) = I_1(\pi - \theta)$. So, if the distribution $I_1(\theta)$ is a polynomial in $\cos\theta$, only the terms with *even* powers of $\cos\theta$ contribute. If we integrate over the angle θ, Eq. (5.46) implies the symmetry: $I_2(\phi) = I_2(\pi - \phi)$. Note that $I_2(\phi)$ is simply the distribution in the Treiman-Yang angle in the Jackson frame. So, parity conservation in the production process means that the Treiman-Yang angle is symmetric around $\phi = \pi/2$. Then, choosing the interval of ϕ between $-\pi/2$ and $3\pi/2$, we may *fold* the distribution in ϕ about $\pi/2$ and consider only the interval between $-\pi/2$ and $+\pi/2$.

If parity is conserved in the decay of the resonance J, we have the additional symmetry, owing to the relations (5.20) and (A.12),

$$I(\theta, \phi) = I(\pi - \theta, \pi + \phi) \ . \tag{5.47}$$

Note that this symmetry is valid, independent of the choice of the quantization axis in the JRF, simply because Eq. (5.47) has been obtained without the use of the symmetry relations of the density matrix. Note also that, if the particles 1 and 2 are identical, we obtain exactly the same symmetry (5.47).

Integrating Eq. (5.47) over the angle θ, we obtain the symmetry: $I_3(\phi) = I_3(\pi + \phi)$. This means that the distribution in ϕ should be symmetric around $\phi = 0$, so that the Treiman-Yang angle distribution in the interval between $-\pi/2$ and $+\pi/2$ may be folded *again* around $\phi = 0$ to give a distribution between 0 and $\pi/2$. Therefore, if parity is conserved both in the production and decay, one may fold the Treiman-Yang angle distribution *twice* in an appropriate way, and consider only the interval between 0 and $\pi/2$, without loss of generality.

5.4 Multipole parameters

Our next task is to define the multipole parameters and expand the angular distribution in these parameters. We shall first define the spherical tensor operators:

$$T_{LM} = \sum_{\Lambda\Lambda'} |J\Lambda\rangle (J\Lambda' LM | J\Lambda) \langle J\Lambda'| \ . \tag{5.48}$$

Using the formula (A.15), we see immediately that under rotations these operators transform according to

$$U[R]T_{LM}U^{+}[R] = \sum_{M'} D^{L}_{M'M}(R)T_{LM'} \ . \tag{5.49}$$

Now, we define the multipole parameter t^J_{LM} for the resonance J as the expectation value of the tensor operators $T_{LM'}$, i.e.

$$t^J_{LM} = tr\{\rho^J T_{LM}\} \ , \tag{5.50}$$

where ρ^J is the density matrix as defined in Eq. (5.30). From Eq. (5.48), we see immediately that

$$t^{J\,*}_{LM} = \sum_{\Lambda\Lambda'} \rho_{\Lambda\Lambda'}(J\Lambda' LM|J\Lambda) \ , \tag{5.51}$$

or, by inverting this,

$$\rho^J_{\Lambda\Lambda'} = \sum_{LM} \left(\frac{2L+1}{2J+1}\right) t^{J\,*}_{LM}(J\Lambda' LM|J\Lambda) \ . \tag{5.52}$$

Then, the density matrix, as defined in Eq. (5.30), may be expressed as

$$\rho^J = \sum_{LM} \left(\frac{2L+1}{2J+1}\right) t^{J\,*}_{LM} T_{LM} \ , \tag{5.53}$$

so that $t^{J\,*}_{LM}$ is simply the coefficient in the expansion of ρ^J in terms of T_{LM}.

From the normalization (5.35), we see that the multipole parameters are normalized so that $T^J_{00} = 1$, while the hermicity of the density matrix implies

$$t^J_{LM} = (-)^M t^{J\,*}_{L-M} \ . \tag{5.54}$$

Note also that $t^J_{LM} = 0$ if $L > 2J$, as is clear from Eq. (5.51). If the z-axis of the JRF lies in the production plane, we have from Eq. (5.41),

$$t^J_{LM} = (-)^{L+M} t^J_{L-M} \ , \tag{5.55}$$

or, by combining with Eq. (5.54),

$$t^{J\,*}_{LM} = (-)^L t^J_{LM} \ . \tag{5.56}$$

On the other hand, if the z-axis of the JRF is along the production normal, we obtain from Eq. (5.45),

$$t^J_{LM} = 0, \qquad \text{for odd } M \ . \tag{5.57}$$

Let us remark at this point that the t^J_{LM}'s may not in general assume arbitrary values but are constrained to a certain physical domain resulting from the positivity of the density matrix and the Eberhard-Good theorem, where applicable. The reader is referred to Jacksom [10] and Byers [16] for simple expressions for the lower and upper bounds of t^J_{LM}; for more elaborate considerations, see Ademollo, Gatto and Preparata [17], and Minnaert [18].

Next, we introduce what we shall call the "moments"; they are the experimental averages of the D-functions:

$$\begin{aligned} H(LM) &= \langle D^L_{M0}(\phi,\theta,0)\rangle \\ &= \int \mathrm{d}\Omega\, I(\Omega)\, D^L_{M0}(\phi,\theta,0)\ . \end{aligned} \tag{5.58}$$

Note that $H(00) = 1$ from Eq. (5.33). Using Eqs. (5.34), (5.51), and (A.16), we find that the moments $H(LM)$ may be expressed as

$$H(LM) = t^{J\,*}_{LM} f^J_L\ , \tag{5.59}$$

where

$$f^J_L = \sum_{\lambda_1\lambda_2} g^J_{\lambda_1\lambda_2}(J\lambda L0|J\lambda),\ \lambda = \lambda_1 - \lambda_2 \tag{5.60}$$

and $f^J_0 = 1$ from Eq. (5.36). So the moment $H(LM)$ is in general given by a product of two terms; the first term $t^{J\,*}_{LM}$ contains the information on how the resonance J is produced, while the second term f^J_L carries the information on the decay of the resonance.

If parity is conserved in the decay, it follows from Eq. (5.20) and the symmetry of the Clebsch-Gordan coefficients that f^J_L satisfies the symmetry

$$f^J_L = 0, \qquad \text{for odd } L\ . \tag{5.61}$$

Note that the same symmetry holds if the two decay products are identical. It is now a simple matter to find the symmetry relations of $H(LM)$; it enjoys all the symmetries that are satisfied by both t^J_{LM} and f^J_L.

The angular distribution has a simple expansion in terms of the moments:

$$I(\Omega) = \sum_{LM}\left(\frac{2L+1}{4\pi}\right) H(LM)\, D^{L\,*}_{M0}(\phi,\theta,0)\ , \tag{5.62}$$

where the sum on L extends from 0 to $2J$. Again, owing to the symmetry (5.54), the angular distribution $I(\Omega)$ is real. For parity-conserving decays, L takes on only the even values in the sum.

6 Three-Particle Systems

A system consisting of three particles may be treated most elegantly in the helicity basis, as was done by Berman and Jacob [19] [for an alternative approach, see Wick [20]]. In this section, we shall first construct a three-particle system in a definite angular momentum state and then apply the formalism to a case of a resonance decaying into three particles. We will give the decay angular distribution in terms of the spin density matrix and discuss the implications of parity conservation. Finally, we will show that in a Dalitz plot analysis different spin-parity states of the three-particle system do not interfere with one another.

Consider a system of three particles 1, 2, and 3. Let us use the notations s_i, η_i, λ_i, and

w_i for the spin, intrinsic parity, helicity, and mass of the particle i. In the rest frame (r.f.) of the three particles, the momentum and energy of the particle i will be denoted by $\vec{p}_i$ and E_i. In the r.f., we define the "standard orientation" of the three-particle system, as shown in Fig. 1.3. this coordinate system is then the "body-fixed" coordinate system, which may be rotated by the Euler angles α, β, and γ to obtain a system with arbitrary orientation.

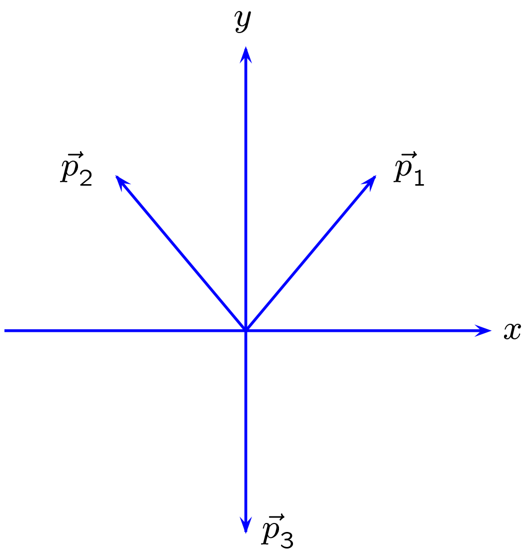

Figure 1.3: Standard orientation of the three-particle rest system. Note that the y-axis is defined along the negative direction of $\vec{p}_3$, and the z-axis along $\vec{p}_1 \times \vec{p}_2$.

A system with the standard orientation can be written

$$|000, E_i\lambda_i\rangle = b \prod_{i=1}^{3} |\vec{p}_i s_i \lambda_i\rangle \ , \tag{6.1}$$

where b is a normalization constant and the helicity basis vectors for each individual particle are given in the usual way [see Eq. (2.16)]:

$$|\vec{p} s_i \lambda_i\rangle = U[R_i L_z(p_i)]\ s_i\lambda_i\rangle \tag{6.2}$$

with

$$R_i = R(\phi_i, \pi/2, 0) \ . \tag{6.3}$$

A three-particle system with an arbitrary orientation in the r.f. can now be obtained by applying a rotation $R(\alpha, \beta, \gamma)$ to the state (6.1):

$$|\alpha\beta\gamma, E_i\lambda_i\rangle = U[R(\alpha, \beta, \gamma)]\, |000, E_i\lambda_i\rangle \ . \tag{6.4}$$

If we impose the normalization of the above states via

$$\langle \alpha'\beta'\gamma', E'_i\lambda'_i|\alpha\beta\gamma, E_i\lambda_i\rangle = \delta^{(3)}(R' - R)\, \delta(E'_1 - E_1)\, \delta(E'_2 - E_2) \prod_i \delta_{\lambda_i\lambda'_i} \tag{6.5}$$

we obtain easily (see Appendix C) that the normalization constant b should be chosen as follows:

$$b^{-1} = 8\pi^2\sqrt{4\pi}\ . \tag{6.6}$$

Let us now define a state of definite angular momentum:

$$|JM\mu, E_i\lambda_i\rangle = \frac{N_J}{\sqrt{2\pi}} \int \mathrm{d}R\, D^{J\,*}_{M\mu}(\alpha,\beta,\gamma)|\alpha\beta\gamma, E_i\lambda_i\rangle\ , \tag{6.7}$$

where N_J is the normalization constant as given in Eq. (4.22). That this state represents a state of definite angular momentum is easy to show following steps identical to those which led to the relation (4.13). Therefore,states (6.7) transform under a rotation R' according to

$$U[R']\,|JM\mu, E_i\lambda_i\rangle = \sum_{M'} D^J_{M'M}(R')|JM'\mu, E_i\lambda_i\rangle\ . \tag{6.8}$$

This relation also shows that, in addition to the obvious invariants E_i and λ_i, the quantity μ is also a rotational invariant.

The physical meaning of μ may be investigated as follows. Let $\vec{n}$ be a unit vector parallel to the body-fixed z-axis, which coincides with the direction $\vec{p}_1\times\vec{p}_2$ in the standard orientation. Now, the integration over $d\gamma$ in Eq. (6.7) involves:

$$\int \mathrm{d}\gamma\, e^{i\mu\gamma}e^{-i\vec{J}\cdot\vec{n}\gamma}|000, E_i\lambda_i\rangle\ .$$

We see that this integration has the effect of picking out from the state $|000, E_i\lambda_i\rangle$ an eigenstate of $\vec{J}\cdot\vec{n}$ with the eigenvalue μ. Then the subsequent rotation by $R(\alpha\beta\gamma)$ [see Eq. (6.4)] makes μ the eigenvalue of $\vec{J}\cdot\vec{n}$ with $\vec{n}$ along the body-fixed z-axis. It now becomes obvious why μ is rotationally invariant; it is the z-component of angular momentum whose quantization axis itself rotates under a rotation of the system.

Let us examine the transformation property of the state (6.7) under parity operations. Since the parity commutes with rotations, we may apply the parity operator $\prod$ on the state (6.1):

$$\begin{aligned}
\Pi|000, E_i\lambda_i\rangle &= b\prod_i \Pi|R_i, p_i, s_i\lambda_i\rangle \\
&= b\prod_i \eta_i e^{-i\pi s_i}|\bar{R}_i, p_i, s_i\, {-\lambda_i}\rangle \\
&= \left\{\prod_i \eta_i e^{-i\pi s_i}\right\} U[R(\pi,0,0)]\,|000, E_i\, {-\lambda_i}\rangle\ ,
\end{aligned} \tag{6.9}$$

where, from Eq. (3.11),

$$\bar{R}_i = R(\pi+\phi_i, \pi/2, 0) = R(\pi,0,0)R_i$$

so that

$$\Pi|\alpha\beta\gamma, E_i\lambda_i\rangle = \left\{\prod_i \eta_i e^{-i\pi s_i}\right\} U[R(\alpha,\beta,\gamma+\pi)]\,|000, E_i\, {-\lambda_i}\rangle\ . \tag{6.10}$$

Using this relation in Eq. (6.7) and changing the integration over γ into one over $\gamma' + \gamma + \pi$, we obtain finally

$$\Pi|JM\mu, E_i\lambda_i\rangle = \eta_1\eta_2\eta_3(-)^{s_1+s_2+s_3-\mu}|JM\mu, E_i \ -\lambda_i\rangle \ . \tag{6.11}$$

We note that this formula is *not* the same as that given in Berman and Jacob [19]. The reason for this is that their definition of one-particle helicity states involves a rotation $R(\phi, \theta, -\phi)$, instead of our convention $R(\phi, \theta, 0)$ [see Eq. (2.16)].

In order to treat the case when two of the three particles are identical, we shall work out a transformation formula for exchanging the particles 1 and 2. The exchange operator $\mathbb{P}_{12}$ applied to the state (6.4) is equivalent to performing a rotation by π around the body-fixed y-axis [see Fig. 1.3]:

$$\mathbb{P}_{12}|\alpha\beta\gamma, E_1\lambda_1, E_2\lambda_2, E_3\lambda_3\rangle = |\pi + \alpha, \pi - \beta, \pi - \gamma, E_2\lambda_2, E_1\lambda_1, E_3\lambda_3\rangle \ , \tag{6.12}$$

where one has used the formula (A.10). Combining this formula with formula (6.7) and using Eq. (A.11),

$$\mathbb{P}_{12}|JM\mu, E_1\lambda_1, E_2\lambda_2, E_3\lambda_3\rangle = (-)^{J+\mu}|JM \ -\mu, E_2\lambda_2, E_1\lambda_1, E_3\lambda_3\rangle \ . \tag{6.13}$$

Again, this formula is not the same as that given in Berman and Jacob [19]. This arises because their standard orientation for the three-particle system has been defined differently from our convention; their coordinate system has been set up with the negative x-axis along the momentum $\vec{p}_3$.

From Eqs. (6.5) and (refeq6.7), we find that our angular momentum states are normalized according to

$$\langle J'M'mu'E_i'\lambda_i'|JM\mu E_i\lambda_i\rangle = \delta_{JJ'}\delta_{MM'}\delta_{\mu\mu'}\delta(E_1 - E_1)\delta(E_2 - E_2)\prod_i \delta_{\lambda_i\lambda_i'} \ . \tag{6.14}$$

The completeness relation is given by

$$\sum_{\substack{JM \\ \mu\lambda_i}} \int |JM\mu E_i\lambda_i\rangle \mathrm{d}E_1\mathrm{d}E_2\langle JM\mu E_i\lambda_i| = I \ . \tag{6.15}$$

From Eqs. (6.5) and (6.7), we obtain the matrix element

$$\langle\alpha\beta\gamma, E_i'\lambda_i'|JM\mu, E_i\lambda_i\rangle = \frac{N_J}{\sqrt{2\pi}} D^{J\,*}_{M\mu}(\alpha, \beta, \gamma)\delta(E_1' - E_1)\delta(E_2' - E_2)\prod_1 \delta_{\lambda_i\lambda_i'} \ . \tag{6.16}$$

We are now ready to discuss the process in which a resonance J with spin-parity η and mass w decays into three particles 1, 2, and 3. In the rest frame of the resonance (JRF), let the angles (α, β, γ) describe the orientation of the three-particle system. Then, the decay amplitude may be written

$$\begin{aligned} A &= \langle\alpha\beta\gamma, E_i\lambda_i|\mathcal{M}|JM\rangle \\ &= \langle\alpha\beta\gamma, E_i\lambda_i|JM\mu E_i\lambda_i\rangle\langle JM\mu E_i\lambda_i|\mathcal{M}|JM\rangle \\ &= \frac{N_J}{\sqrt{2\pi}} F^J_\mu(E_i\lambda_i) D^{J\,*}_{M\mu}(\alpha\beta\gamma) \end{aligned} \tag{6.17}$$

after using formulae (6.15) and (6.16). If the "decay operator" $\mathcal{M}$ is rotationally invariant, the decay amplitude F should depend *only* on the rotational invariants, i.e.

$$F^j_\mu(E_i\lambda_i) = \langle JM\mu E_i\lambda_i|\mathcal{M}|JM\rangle \; . \tag{6.18}$$

If parity is conserved in the decay, we have the symmetry from Eq. (6.11):

$$F^J_\mu(E_i\lambda_i) = \eta\eta_1\eta_2\eta_3(-)^{s_1+s_2+s_3+\mu}F^j_\mu(E_i\,-\lambda_i) \; . \tag{6.19}$$

And, if particles 1 and 2 are identical,

$$F^J_\mu(E_1\lambda_1, E_2\lambda_2, E_3\lambda_3) = \pm(-)^{J+\mu}F^J_{-\mu}(E_2\lambda_2, E_1\lambda_1, E_3\lambda_3) \; , \tag{6.20}$$

where the plus sign holds for two identical bosons and the minus sign for fermions.

Let us assume that the resonance J is produced in the following process:

$$a + b \to c + J, \quad J + 1 + 2 + 3 \; . \tag{6.21}$$

In analogy to the two-body decays, we introduce the density matrix for the resonance J, and assume as before that it is independent of w, the resonance mass. From Eq. (6.17), we may write the differential cross-section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}R\mathrm{d}w\mathrm{d}E_1\mathrm{d}E_2} \sim \left(\frac{2J+1}{8\pi^2}\right)\sum_{\substack{MM'\\ \mu\mu'}}\rho^J_{MM'}D^{J\,*}_{M\mu}(R)D^J_{M'\mu'}(R)\times K(w)\sum_{\lambda_i}F^J_\mu F^{J\,*}_{\mu'} \; , \tag{6.22}$$

where $R = R(\alpha,\beta,\gamma)$ is the rotation specifying the orientation of the three-particle system in the JRF, and $K(w)$ is the kinematic factor which contains, among others, the phase space factors [see formula (B.11)].

If we integrate over $d\gamma$, dE_1, dE_2, and dw, we obtain the angular distribution in $\Omega = (\theta,\phi)$ describing the direction of the normal to the decay plane:

$$I(\Omega) = \left(\frac{2J+1}{4\pi}\right)\sum_{MM'}\sum_{\mu}D^{J\,*}_{m_\mu}(\phi,\theta,0)g^J_\mu \tag{6.23}$$

where

$$g^J_\mu = \int \mathrm{d}w\,\mathrm{d}E_1\,\mathrm{d}E_2\,K(w)\sum_{\lambda_i}|F^J_\mu(E_i\lambda_i)|^2 \; . \tag{6.24}$$

Note that $I(\Omega)$ is properly normalized, if we require that $\Sigma_M\rho^J_{MM} = 1$ and $\Sigma_\mu g^J_\mu = 1$.

If the z-axis in the JRF is fixed to be i the production plane, we obtain the symmetry, following the same argument as that for the two-body decays,

$$I(\theta,\phi) = I(\pi+\theta,\pi-\phi) \; . \tag{6.25}$$

This is identical to formula (5.46). So, in the Jackson frame, the distribution in the Treiman-Yang angle may be folded around $\pi/2$, and one may consider only the interval between $-\pi/2$ and $+\pi/2$.

The distribution $I(\Omega)$ for three-particle decays is different in *one* important aspect from that for two-body decays: the parity conservation in the decay process *does not* in general lead to any additional symmetry in $I(\Omega)$, as g^J_μ remains invariant under parity [see Eqs. (6.19) and (6.24)]. One important exception occurs in the case of the decay $\omega \to 3\pi$. The parity conservation implies from Eq. (6.19) that $g^1_{\pm 1} = 0$ with only one non-zero component g^1_0, so that the resulting angular distribution is identical to a two-body decay, i.e. $\rho \to 2\pi$ [compare expressions (5.34) and (6.23)]. In this case, then, we have the additional symmetry given by Eq. (5.47).

It is possible to obtain an additional symmetry, if the particles 1 and 2 are identical. From formulae (6.20) and (6.24), we see that $g^J_\mu = g^J_{-\mu}$ in this case. Then, using expressions (6.23) and (A.12), we get

$$I(\theta, \phi) = I(\pi - \theta, \pi + \phi) \tag{6.26}$$

identical to Eq. (5.47) for two-body decays. Following the same argument in Section 5.3, we conclude that the Treiman-Yang angle distribution can be confined to the interval between 0 and $+\pi/2$.

The angular distribution of Eq. (6.23) may be expanded in terms of the moments $H(LM)$ in the same way as in Section 5.4. The relations (5.58), (5.59), and (5.62) remain the same, and the f^J_L of Eq. (5.60) is now given by

$$f^J_L = \sum_\mu g^J_\mu (J\mu L0|J\mu) \ . \tag{6.27}$$

If the particles 1 and 2 are identical $f^J_L = 0$ for odd L, as was the case for the two-body decays.

As a final item to be discussed in this section, we shall show that, in a Dalitz plot analysis, two different spin-parity states do not interfere with each other. Suppose that in a reaction two resonances are produced with spins J_1 and J_2, each of which in turn decays into a common set of three particles. The over-all amplitude may be written

$$m_{fi} \sim \sum_{M_1\mu_1} T_1(M_1) D^{J_1\,*}_{M_1\mu_1}(R) F^{J_1}_{\mu_1}(E_i\lambda_i) + \sum_{M_2\mu_2} T_2(M_2) D^{J_2\,*}_{M_2\mu_2}(R) F^{J_2}_{\mu_2}(E_i\lambda_i) \ . \tag{6.28}$$

where $T_i(M)$ is the production amplitude. If $J_1 \neq J_2$, the Dalitz plot distribution is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}w\mathrm{d}E_1\mathrm{d}E_2} \sim \sum_{M_1\mu_1\lambda_i} |T_1(M_1) F^{J_1}_{\mu_1}|^2 + \sum_{M_2\mu_2\lambda_i} |T_2(M_2) F^{J_2}_{\mu_2}|^2 \tag{6.29}$$

after integrating over dR. This shows that, if one integrates over the orientation of the three-particle system, states of two different spins do not interfere due to the orthogonality of the D-functions.

Suppose now that the spins are the same but the parities of the two resonances are opposite, Again, integrating over dR, one obtains

$$\frac{\mathrm{d}\sigma}{\mathrm{d}w\mathrm{d}E_1\mathrm{d}E_2} \sim \sum_{M\mu\lambda_i} |T_1(M) F^J_\mu + T_2(M) \bar{F}^J_\mu|^2 \ , \tag{6.30}$$

where $\bar{F}$ indicates a decay amplitude of opposite parity to that of F. Applying the symmetry (6.19) from parity conservation, one may rewrite expression (6.30) with a minus sign in front of the second term, which means that the interference term is identically zero, again obtaining the result (6.29). In conclusion, we may state that, as long as one integrates over the orientation of the three particle system and sums over the helicities of the final particles, states of different spin-parity do not interfere with one another in a Dalitz-plot analysis.

7 Sequential Decays

If a resonance decays in two steps, each consisting of a two-body decay, then the moments obtained from the joint decay distribution provide a powerful means of determining the spin and parity of the parent resonance. In this section we shall develop a general formalism for the sequential decay, $J \to s + 0$, $s \to s_1 + 0$, where the spins J, s and s_1 are arbitrary, and illustrate the formalism with a few simple but, i practice, important examples, namely $\Sigma(1385) \to \Lambda + \pi$, $\Delta(1950) \to \Delta(1232) + \pi$, $b_1(1235) \to \pi + \omega$, and $\pi_2(1670) \to \pi + f_2(1270)$.

It is possible to use for the spin-parity analysis two-particle states constructed from the canonical basis vectors, as was done by Ademollo and Gatto [21] and Ademollo, Gatto and Preparate [22]. We shall adopt, however, the helicity formalism in this section, for it not only involves simpler algebra but also brings out certain salient features in the problem, not quite transparent within the canonical formalism. The helicity formalism was first used by Byers and Fenster [23], who treated the case of a resonance decaying into a $\Lambda + \pi$ system with $\Lambda \to p + \pi^-$. Their method has been successfully employed [24] to determine the spin and parity of $\Xi(1530)$ ad $\Sigma(1385)$. Button-Shafer [25] later has extended the method to treat a fermion resonance decaying into a pins-3/2 baryon and a pion, and chung[15] has applied the technique to treat a boson resonance into two intermediate bosons with spin, each of which in turn decays into two or three spinless particles. Ascoli *et al.* [26] have used a formalism very similar to that of Chung in their spin-parity analysis of the $b_1(1235)$ meson. Berman and Jacob [27] have also given a similar formalism treating both the fermion and boson resonances. Donohue [28], on the other hand, treats the problem of analyzing a boson resonance decaying into a fermion and an anti-fermion (see also Ref. [22]).

Let us suppose that a resonance of unknown spin-parity J^η with mass w is produced and decays via the following chain of processes:

$$\begin{aligned} a + b &\rightarrow c + J \\ J &\rightarrow s + \pi_1 \\ s &\rightarrow s_1 + \pi_2 \ , \end{aligned} \tag{7.1}$$

where π_1 or π_2 stands for pion and J, s, and s_1 designate the parent, intermediate, and final particles, as well as their spin. Let η_s, λ, and w_s (η_1, λ_1, and w_1) denote the parity, helicity and mass of the particle s (s_1). In the rest frame of J (s), denoted by JRF (sRF), let P_s (p_1) and Ω (Ω_1) be the magnitude and direction of the s (s_1) momentum measured in the helicity coordinate system as shown in Fig. 1.1b.

Although the sRF should always be described by the helicity coordinate system, which is related to the JRF as illustrated in Fig. 1.1b, the coordinate axes specifying the JRF need not be those of the helicity system with respect to the production coordinate axes; the JRF can just as well be described by the Jackson coordinate system, or the one with the z-axis along the production normal (see Section 5.3). For concreteness, however, we shall use the helicity coordinate system for the JRF, and use the symbol Λ for the helicity of the resonance J, bearing in mind that we are at liberty to choose any coordinate system we wish for the JRF.

The over-all invariant amplitude for the process (7.1) may be written

$$\mathcal{M}_{fi} \sim \langle \Omega_1 s_1 \lambda_1 | \mathcal{M}_s | s\lambda \rangle \; \langle \Omega s \lambda | \mathcal{M}_J | J\Lambda \rangle \; \langle \vec{p}_f \lambda_c \Lambda | T(w_0) | \vec{p} \lambda_a \lambda_b \rangle \;, \tag{7.2}$$

where the first and the second factor describe the s and J decay, respectively, and the third factor is the production amplitude for the J with helicity Λ. This production amplitude is identical to that given in (5.26).

As in Section 5.3, we make the simplifying assumption that the J production amplitude is independent of w. Then, the spin density matrix for the J as given in Eq. (5.28) is a constant and can safely be normalized according to Eq. (5.35). The decay amplitudes in Eq. (7.2) are given, according to Eq. (5.16), by

$$\langle \Omega s \lambda | \mathcal{M}_J | J\Lambda \rangle = N_J F^J_\lambda \, D^{J\,*}_{\Lambda\lambda}(\phi, \theta, 0) \tag{7.3}$$

and

$$\langle \Omega_1 s_1 \lambda_1 | \mathcal{M}_s | s\lambda \rangle = N_s F^s_{\lambda_1} \, D^{s\,*}_{\lambda\lambda_1}(\phi_1, \theta_1, 0) \;, \tag{7.4}$$

where $\Omega = (\theta, \phi)$ and $\Omega_1 = (\theta_1, \phi_1)$. In analogy to Eq. (5.31), we shall introduce the following symbols to describe the bilinear products of the helicity amplitudes:

$$g^J_{\lambda\lambda'} = \int \mathrm{d}w \, \mathrm{d}w_s K(w, w_s) F^J_\lambda F^{J\,*}_{\lambda'} \tag{7.5}$$

and

$$g^s_{\lambda_1} = |F^s_{\lambda_1}|^2 \;, \tag{7.6}$$

where $K(w, w_s)$ includes all the functions of w or w_s, such as the squares of Breit-Wigner functions for the particles J and s and the phase-space factors [see Eq. (B.10)].

We emphasize that any dependence of w_s in the helicity amplitude $F^s_{\lambda_1}$ has been factored out and absorbed into $K(w, w_s)$, so that $g^s_{\lambda_1}$ can be considered constant. As pointed out in Section 5.3, this is not in general possible. However, with the examples we consider here, this is an excellent approximation: if the intermediate resonance s is $\Lambda(1115)$, the helicity amplitude $F^s_{\lambda_1}$ is clearly a constant, owing to the narrow width of the Λ; if the s is $\Delta(1232)$, $F^s_{\lambda_1}$ may be assumed to be proportional to p_1, owing to the p-wave nature of the decay, and the p_1^2 factor may then be absorbed into $K(w, w_s)$.

Now, we are ready to write down the joint angular distribution in Ω and Ω_1:

$$\begin{aligned} I(\Omega, \Omega_1) = \left(\frac{2J+1}{4\pi}\right) \left(\frac{2s+1}{4\pi}\right) &\sum_{\substack{\Lambda\Lambda' \\ \lambda\lambda'\lambda_1}} \rho^J_{\Lambda\Lambda'} \, g^J_{\lambda\lambda'} \, g^s_{\lambda_1} \\ &\times D^{J\,*}_{\Lambda\lambda}(\Omega) D^J_{\Lambda'\lambda'}(\Omega) D^{s\,*}_{\lambda\lambda_1}(\Omega_1) D^s_{\lambda'\lambda_1}(\Omega_1) \;, \end{aligned} \tag{7.7}$$

where we have used the shorthand notation:

$$D^J_{\mu m}(\Omega) = D^J_{\mu m}(\phi, \theta, 0)\ . \tag{7.8}$$

We shall adopt the following normalizations for the g's:

$$\sum_\lambda g^J_{\lambda\lambda} \;=\; 1 \tag{7.9}$$

$$\sum_{\lambda_1} g^s_{\lambda_1} \;=\; 1\ , \tag{7.10}$$

so that the joint angular distribution is normalized according to

$$\int \mathrm{d}\Omega\, \mathrm{d}\Omega_1 I(\Omega, \Omega_1) = 1 \tag{7.11}$$

with the trace of ρ^J equal to 1 as given in Eq. (5.35).

Let us now introduce the joint "moments", which are the experimental averages of the product of two D-functions:

$$H(\ell m L M) = \langle D^L_{M\,m}(\Omega) D^\ell_{m\,0}(\Omega_1)\rangle \tag{7.12}$$

with the normalization,

$$H(0000) = 1\ . \tag{7.13}$$

Then the moments are given by

$$H(\ell m L M) = \int \mathrm{d}\Omega\, \mathrm{d}\Omega_1\, I(\Omega, \Omega_1) D^L_{Mm}(\Omega) D^\ell_{m0}(\Omega_1)\ . \tag{7.14}$$

Using Eqs. (7.7) and (A.16), we find that the H's may be expressed as:

$$H(\ell m L M) = t^{J\,*}_{LM} f^{JL}_{\ell m} f^s_\ell\ , \tag{7.15}$$

where $t^{J\,*}_{LM}$ is the multipole parameter as given in Eqs. (5.51), and the f's are related to the g's by

$$f^{JL}_{\ell m} = \sum_{\lambda\lambda'} g^J_{\lambda\lambda'} (J\lambda' L m|J\lambda)(s\lambda' \ell m|s\lambda) \tag{7.16}$$

and

$$f^s_\ell = \sum_{\lambda_1} g^s_{\lambda_1} (s\lambda_1 \ell 0|s\lambda_1) \tag{7.17}$$

with the normalizations given by

$$f^{J0}_{00} = f^s_0 = 1\ . \tag{7.18}$$

The expression (7.15) displays neatly how the moments H depend on the resonances J and s. The H's are in general a product of three factors; the first one carries the information on how the resonance J has been produced, while the second (third) one contains the information on the decay of the resonance J (resonance s).

Let us turn to a discussion of the symmetry relations satisfied by the f's. For this purpose, we first recall that parity conservation in the decay implies, according to Eq. (5.20),

$$F^J_\lambda = \varepsilon F^J_{-\lambda}, \qquad \varepsilon = \eta\eta_s(-)^{J-s+1} \tag{7.19}$$

and

$$F^s_{\lambda_1} = \varepsilon_s F^s_{-\lambda_1}, \qquad \varepsilon_s = \eta_s\eta_1(-)^{s-s_1+1} \tag{7.20}$$

so that we have the conditions

$$g^J_{\lambda\lambda'} = \varepsilon g^J_{-\lambda\lambda'} = \varepsilon g^J_{\lambda-\lambda'} = g^J_{-\lambda-\lambda'} \ . \tag{7.21a}$$

Also, if time-reversal invariance is applicable, we should have [see Eq. (5.24)],

$$g^J_{\lambda\lambda'} \simeq \text{real} \tag{7.21b}$$

where the symbol $\simeq$ reminds us that this symmetry may *not* hold in all cases. From Eq. (7.20),

$$g^s_{\lambda_1} = g^s_{-\lambda_1} \ . \tag{7.22}$$

In addition, by the definition (7.5),

$$g^J_{\lambda\lambda'} = g^{J\,*}_{\lambda'\lambda} \ . \tag{7.23}$$

Now, the symmetry relations on the f's can be derived easily using the definition (7.16) and Eq. (7.23),

$$f^{JL}_{\ell-m} = f^{JL\,*}_{\ell m} \ . \tag{7.24a}$$

From parity conservation [using Eq. (7.21a)]

$$f^{JL}_{\ell m} = (-)^{\ell+L} f^J_{\ell-m} \ . \tag{7.24b}$$

Or, by combining the above two relations,

$$f^{JL}_{\ell m} = (-)^{\ell+L} f^{JL\,*}_{\ell m} \ . \tag{7.24c}$$

From Eq. (7.21b),

$$f^{JL}_{\ell m} \simeq \text{real} \tag{7.24d}$$

or, from Eq. (7.24c),

$$f^{JL}_{\ell m} \simeq 0, \qquad \text{for odd } (\ell + L) \ . \tag{7.24e}$$

If the s decay is parity-conserving, one has

$$f^s_\ell = 0, \qquad \text{for odd } \ell \ . \tag{7.25}$$

The symmetry relations on the H's are readily derived, once the symmetry relations on the multipole parameters and on the two f's are known. Firs of all, from the definition (7.12) and Eq. (A.9), we have

$$H^*(\ell m L M) = (-)^M H(\ell\ -m\ L\ -M) \ . \tag{7.26a}$$

Also, from the definitions (5.51), (7.16) and (7.17),

$$J(\ell m L M) \ = 0, \qquad \text{if } \ell > 2s \text{ or } L > 2J \ . \tag{7.26b}$$

From parity conservation in the J decay, using Eq. (7.24b),

$$H(\ell m L M) \ = (-)^{\ell+L} H(\ell - m L M) \ . \tag{7.26c}$$

If the s decay is parity-conserving, from Eq. (7.25),

$$H(\ell m L M) \ = 0, \qquad \text{for odd } \ell \ . \tag{7.26d}$$

If the argument for time-reversal invariance as given in Section 5.2 is applicable, from Eq. (7.24e),

$$H(\ell m L M) \ \simeq 0, \qquad \text{for odd } (\ell + L) \tag{7.26e}$$

Finally, parity conservation in the production process leads to the following additional symmetry. If the JRF z-axis is in the production plane, from Eq. (5.55),

$$H(\ell m L M) \ = (-)^{L+M} H(\ell m L - M) \ . \tag{7.26f}$$

If the z-axis is along the production normal,

$$H(\ell m L M) \ = 0, \qquad \text{for odd } M \ . \tag{7.26g}$$

For the spin-parity analysis, it turns out to be useful to invert Eq. (7.16) using the orthonormality of the Clebsch-Gordan coefficients:

$$g^J_{\lambda\lambda'}(J\lambda' L m | J\lambda) = \sum_\ell \left(\frac{2\ell+1}{2s+1}\right) f^{JL}_{\ell m}(s\lambda'\ell m | s\lambda) \ . \tag{7.27}$$

Multiplying both sides by $t^{J\,*}_{LM}$, one obtains

$$t^{J\,*}_{LM}\, g^J_{\lambda\lambda'}(J\lambda' L m | J\lambda) = G_{\lambda\lambda'}(LM) \ , \tag{7.28}$$

where

$$G_{\lambda\lambda'}(LM) = \sum_\ell \left(\frac{2\ell+1}{2s+1}\right) (s\lambda'\ell m | s\lambda)(f^s_\ell)^{-1} H(\ell m L M) \ . \tag{7.29}$$

We shall see, in the examples to be given later, that f^s_ℓ is always a know quantity, being proportional to a Clebsch-Gordan coefficient. Therefore, the $G_{\lambda\lambda'}(LM)$ is an experimentally measurable quantity; it is this quantity that yields most directly the information on the spin and parity of the resonance J.

Because of the symmetry (7.26d), it is not possible to perform the sum on ℓ in Eq. (7.29) for odd ℓ, if the s decay is parity-conserving. It is easy to restrict, however, the sum on ℓ to even values by the following procedure: exchange the index λ to $-\lambda'$ and λ' to $-\lambda$ in Eq. (7.27), and add the resulting equation to the original formula (7.27), to obtain

$$t^{J\,*}_{LM}\ \frac{1}{2}[\, g^J_{\lambda\lambda'} + (-)^L g^{J\,*}_{\lambda\lambda'}\,]\ (J\lambda' L m | J\lambda) = G^{(+)}_{\lambda\lambda'}(LM) \ , \tag{7.30}$$

where the $G^{(+)}_{\lambda\lambda'}(LM)$ now has the same form as the $G_{\lambda\lambda'}(LM)$ of Eq. (7.29) but the sum is confined to even values of ℓ. Therefore, if the parity is conserved in the decay of the intermediate resonance s, it is the $G^{(+)}_{\lambda\lambda'}(LM)$ that is experimentally measurable. We shall see later that measurement of the $G^{(+)}_{\lambda\lambda'}(LM)$'s for all allowed values of λ, λ', L and M enables us to determine uniquely the spin and parity of the J.

Let us note the following symmetry relations satisfied by the $G^{(+)}$'s: from Eq. (7.26c),

$$G^{(+)}_{\lambda'\lambda}(LM) \ = (-)^{L+\lambda-\lambda'} G^{(+)}_{\lambda\lambda'}(LM) \tag{7.31a}$$

and

$$G^{(+)}_{-\lambda-\lambda'}(LM) \ = (-)^{L} G^{(+)}_{\lambda\lambda'}(LM) \ . \tag{7.31b}$$

In addition, the $G^{(+)}$'s clearly obey the same symmetries as those satisfied by the multipole parameters $t^{J\,*}_{LM}$.

At this point, we shall briefly mention how we can test experimentally the applicability of the time-reversal invariance discussed in Section 5.2. It is clear from Eq. (7.26e) that, if the H's are non-zero for odd $(\ell + L)$, the argument of Section 5.2 is not applicable; or, equivalently, one may look for non-zero $G^{(+)}_{\lambda\lambda'}(LM)$ with odd L, since it should be zero if the H's are zero for odd $(\ell + L)$. From the point of view of the spin-parity analysis, we may state that, in general, the H's with odd $(\ell + L)$ or the $G^{(+)}$'s with odd L may not be as useful in yielding the quantum numbers of the J as the other H's or $G^{(+)}$'s, for they may be either zero or approximately zero.

In analogy to the two-body decays discussed in Section 5.4, the joint angular distribution given in Eq. (7.7) has a simple expansion in terms of the product of two D-functions:

$$I(\Omega, \Omega_1) = \sum_{\substack{\ell m \\ LM}} \left(\frac{2\ell+1}{4\pi}\right)\left(\frac{2L+1}{4\pi}\right) H(\ell m L M) D^{L\,*}_{Mm}(\Omega) D^{\ell\,*}_{m0}(\Omega_1) \ . \tag{7.32}$$

Note that this angular distribution is, of course, real, owing to the symmetry (7.26a) for the H's. The formula (7.32) affords an alternative method of determining the moments H; by using the maximum likelihood method, one may fit the joint angular distribution with the formula (7.32), using the H's as the unknown parameters after taking into account the symmetries (7.26a)–(7.26g). One experimental uncertainty is, of course, how large a number one may take as the maximum value of L ($L_{\max}$). Perhaps the most reasonable procedure is to take the smallest $L_{\max}$ for which an acceptable fit to the data can be obtained.

In the remainder of this section, we shall apply the general formalism developed so far to a few concrete and, in practice, important examples. It is hoped that the examples selected here are sufficiently diverse to give the reader an over-all picture of the various techniques involved and that, after going through these examples, he has acquired enough skill with which he can tackle any new decay modes he may encounter. In fact, our formalism can be easily generalized to treat resonances with decay products which both have spin, for example $\Lambda + \omega$ or $\rho + \omega$.

7.1 $\Sigma(1385) \to \Lambda + \pi$

In this case, the intermediate particle s is the $\Lambda(1115)$ with $s^{\eta_s} = \frac{1}{2}^+$, which decays into the $(p + \pi^-)$ system via the weak interaction. Note that the final particle s_1 is the proton, i.e. $s_1^{\eta_1} = \frac{1}{2}^+$. We wish to determine the spin-parity J^η of $\Sigma(1385)$ by the moment analysis.

Let us first write the normalization (7.9) explicitly:

$$g^J_{++} + g^J_{--} = 2g^J_{++} = 1$$

so that

$$\begin{aligned} g^J_{++} &= g^J_{--} = 1/2 \\ g^J_{+-} &= g^J_{-+} = \varepsilon/2 \,, \end{aligned} \tag{7.33}$$

where $\varepsilon = \eta(-)^{J+\frac{1}{2}}$. Since the Λ decay is parity non-conserving, f^s_ℓ is non-zero for both $\ell = 0$ and 1:

$$\begin{aligned} f^s_0 &= g^s_+ + g^s_- = 1 \\ f^s_1 &= (g^s_+ - g^s_-)(\tfrac{1}{2}\tfrac{1}{2}10|\tfrac{1}{2}\tfrac{1}{2}) \,. \end{aligned} \tag{7.34}$$

We do not give the explicit expression for $g^s_{\lambda_1}$, but is related to the well-known decay asymmetry parameter α, so that f^s_1 may be considered a known quantity.

Let us write down explicitly the G's using Eq. (7.29). For arbitrary values of L and M,

$$\begin{aligned} G_{++}(LM) &= \frac{1}{2}H(00LM) + \frac{3}{2}(\tfrac{1}{2}\tfrac{1}{2}10|\tfrac{1}{2}\tfrac{1}{2})(f^s_1)^{-1}H(10LM) \\ &= \frac{1}{2}H(00LM), \qquad \text{for even } L \\ &= \frac{\sqrt{3}}{2}(f^s_1)^{-1}H(00LM), \qquad \text{for odd } L \,, \end{aligned} \tag{7.35}$$

where one has used the symmetry (7.26c). Also,

$$\begin{aligned} G_{+-}(LM) &= \frac{3}{2}(\tfrac{1}{2} - \tfrac{1}{2}11|\tfrac{1}{2}\tfrac{1}{2})(f^s_1)^{-1}H(11LM) \\ &= -\frac{\sqrt{3}}{2}(f^s_1)^{-1}H(11LM) \qquad (\text{odd } L) \,. \end{aligned} \tag{7.36}$$

Using the relations (7.28) and (7.33), one obtains for the ratio of Eqs. (7.36) to (7.35),

$$\frac{G_{+-}(LM)}{G_{++}(LM)} = -\sqrt{2}\frac{H(11LM)}{H(10LM)} = \varepsilon\frac{(J - \frac{1}{2}L1|J\frac{1}{2})}{(J\frac{1}{2}L0|J\frac{1}{2})} \qquad (\text{odd } L) \,.$$

From Eq. (A.17), one obtains the final result

$$\frac{H(11LM)}{H(10LM)} = \varepsilon\frac{2J+1}{\sqrt{2L(L+1)}} \qquad (\text{odd } L) \,. \tag{7.37}$$

The formula (7.37) may be used to determine both the spin and parity of the resonance J: the spin can be determined by evaluating the absolute value of the ratio of the two moments, while the parity can be determined by the relative sign of the two moments. This method has been used to determine the spin and parity of $\Sigma(1385)$. One must exercise some care in using the formula (7.37), for the ratio of two experimental averages does not have a Gaussian distribution, even though the averages themselves may be Gaussian.

A simpler approach might be to write formula (7.37) in the form

$$H(11LM) - \varepsilon\frac{2J+1}{\sqrt{2L(L+1)}}H(10LM) = 0 \qquad \text{(odd } L) \tag{7.38}$$

and, for given values of L and M, test this equality for all spin-parity combinations. In this way, one can obtain the χ^2 probability for each possible spin-parity assignment. Of course, it is necessary to choose the values of L and M such that both the moments in Eq. (7.38) are appreciably different from zero (or different from the background). Otherwise, the formula will give *no* differentiation between different spin-parity assignments.

Note that, although the formula (7.37) or (7.38) can be used for any odd L, it is most useful for $L+1$, for then it is applicable to any spin $H \geq \frac{1}{2}$ [see Eq. (7.26b)]. If the spin J turns out to be greater that $\frac{1}{2}$, one may use higher odd L as a consistency check.

7.2 $\Delta(1950) \to \Delta(1232) + \pi$

We shall now treat the case where the intermediate resonance s decays via the parity-conserving strong interaction. Specifically, we shall consider a high-mass isobar decaying into $\Delta(1232)+\pi$. Note that in this case $s^{\eta_s} = 3/2^+$ for the Δ and $s_1^{\eta_1} = 1/2^+$ for the nucleon.

Let us first note that, because of the normalization (7.10) and the parity conservation in the $\Delta(1232)$ decay, we have

$$g_+^s = g_-^s = \frac{1}{2}\ . \tag{7.39}$$

Then, from Eq. (7.17), we see that

$$f_\ell^s = 0, \quad \text{for odd } \ell; \quad f_\ell^s = (\tfrac{3}{2}\tfrac{1}{2}\ell 0|\tfrac{3}{2}\tfrac{1}{2}), \quad \text{for even } \ell\ . \tag{7.40}$$

The $G^{(+)}$'s are now experimentally accessible quantities given by

$$G^{(+)}_{\lambda\lambda'}(LM) + \sum_{\ell=0,2}\left(\frac{2\ell+1}{4}\right)\frac{(\frac{3}{2}\lambda'\ell m|\frac{3}{2}\lambda)}{(\frac{3}{2}\frac{1}{2}\ell 0|\frac{3}{2}\frac{1}{2})}H(\ell m LM)\ . \tag{7.41}$$

Explicitly written, the $G^{(+)}$'s have the form

$$\begin{aligned}
G^{(+)}_{\frac{1}{2}-\frac{1}{2}}(LM) &= G^{(+)}_{\frac{3}{2}-\frac{3}{2}}(LM) = 0 && && (7.42a)\\
G^{(+)}_{\frac{1}{2}-\frac{3}{2}}(LM) &= -\left(\frac{5\sqrt{2}}{4}\right)H(22LM) && (L \geq 2) && (7.42b)\\
G^{(+)}_{\frac{1}{2}-\frac{3}{2}}(LM) &= \left(\frac{5\sqrt{2}}{4}\right)H21LM) && (L \geq 1) && (7.42c)\\
G^{(+)}_{\frac{1}{2}\frac{1}{2}}(LM) &= \frac{1}{4}H(00LM) + \frac{5}{4}H(20LM) && \text{(even } L) && (7.42d)\\
G^{(+)}_{\frac{3}{2}\frac{3}{2}}(LM) &= \frac{1}{4}H(00LM) + \frac{5}{4}H(20LM) && \text{(even } L)\ . && (7.42e)
\end{aligned}$$

The last two relations can be used to determine the following ratio, using the formula (7.30),

$$\begin{aligned}\frac{g^J_{\frac{3}{2}\frac{3}{2}}}{g^J_{\frac{1}{2}\frac{1}{2}}} &= \frac{G^{(+)}_{\frac{3}{2}\frac{3}{2}}(00)}{G^{(+)}_{\frac{1}{2}\frac{1}{2}}(00)} \\ &= \frac{1-5H(2000)}{1+5H(2000)} \; . \end{aligned} \tag{7.43}$$

This allows us to write down the following spin formula, applicable if $J \geq 3/2$:

$$\left(\frac{g^J_{\frac{1}{2}\frac{1}{2}}}{g^J_{\frac{3}{2}\frac{3}{2}}}\right) \frac{G^{(+)}_{\frac{3}{2}\frac{3}{2}}(LM)}{G^{(+)}_{\frac{1}{2}\frac{1}{2}}(LM)} = \frac{J\frac{3}{2}L0|J\frac{3}{2})}{(J\frac{1}{2}L0|J\frac{1}{2})} \qquad (\text{even } L \geq 2) \; .$$

Or, more explicitly, by using Eq. (A.18),

$$\left[\frac{1+5H(2000)}{1-5H(2000)}\right]\left[\frac{H(00LM)-5H(20LM)}{H(00LM)+5H(20LM)}\right] = 1 - \frac{4L(L+1)}{4J(J+1)-3} \qquad (\text{even } L \geq 2) \; . \tag{7.44}$$

This formula is a potentially powerful one; it allows one to determine uniquely the spin $J \geq 3/2$ regardless of the parity of the parent resonance, if the relevant moments are found to be non-zero for $L = 2$ and some allowed M.

It should be emphasized that we have been able to obtain the formula (7.44) as a consequence of our assumption that the density matrix for the J does not depend on the invariant mass w of the J. If this assumption is relaxed, both the t^J_{LM} and $g^J_{\lambda\lambda'}$ are in general functions of w and they cannot be separated from each other and treated as constants, once the integration over w has been performed over the region of the resonance J.

There is yet another formula by which one can determine simultaneously the spin and parity of the parent resonance. Let us take the ratio of Eqs. (7.42*b*) to (7.42*c*) and utilize the relations (7.30) and (A.19):

$$\frac{H(22LM)}{H(21LM)} = \varepsilon \frac{J+\frac{1}{2}}{[(L-1)(L+2)]^{\frac{1}{2}}} \qquad (\text{even } L \geq 2) \; , \tag{7.45}$$

where $\varepsilon = \eta(-)^{J-\frac{1}{2}}$. Note that, again, the above formula can be used only if $J \geq 3/2$ [see Eq. (7.26*b*)].

Let us now consider the case when $J = 1/2$. From Eq. (7.26*b*), we see immediately that

$$H(\ell mLM) = 0 \qquad \text{for all } L > 1 \; . \tag{7.46}$$

Moreover, $G^{(+)}_{\lambda\lambda'}$ should be zero if either λ or λ' is equal to $\pm 3/2$. Then, from Eqs. (7.42*a*)–(7.42*e*),

$$H(21LM) = 0 \qquad (L = 1) \tag{7.47}$$

and

$$H(2000) = \frac{1}{5} \; . \tag{7.48}$$

The relation (7.47) might not be a useful test, because of the possibility that it could be a consequence of time-reversal invariance.

It is rather unfortunate that the parity cannot be determined within our formalism if $J = 1/2$. One can determine the parity, however, if the fermion s_1 is unstable and its decay angles are also observed [see Button-Shafer [25]]. An example might be the following decay sequence: $\Xi(1820) \to \Xi(1530) + \pi$, $\Xi(1530) \to \Xi + \pi$, and $\Xi \to \Lambda + \pi$.

7.3 $b_1(1235) \to \pi + \omega$

As pointed out in Section 6, the decay of the ω can be treated on an identical footing to that of the ρ^0 within our formalism; we merely have to use the normal to the decay plane of ω as the analyzer instead of the relative momentum of the ρ^0 decay. We take up the $\pi\omega$ decay mode as our first example of the spin-parity analysis of boson resonances, because, owing to the narrow width of ω, there is negligible interference effect coming from the identity of two pions in the 4π final state. Thus, this decay mode constitutes an ideal case for our moment analysis. This problem has been treated using various techniques by Zemach [29], Ademollo, Gatto and Preparata [30], Berman and Jacob [27], and Chung [15].

Before we proceed to discuss the $\pi\omega$ decay mode, we shall write down the form of the $G^{(+)}$'s valid for any arbitrary intermediate resonance s decaying into two pions. In this case, we have $s_1^{\eta_1} = 0^-$ and $g^s = 1$, so that

$$f_\ell^s = (s0\ell0|s0) \ . \tag{7.49}$$

From Eq. (7.29), we obtain the formula

$$G_{\lambda\lambda'}^{(+)}(LM) = \sum_{\text{even } \ell} \left(\frac{2\ell+1}{2s+1}\right) \frac{(s\lambda'\ell m|s\lambda)}{(s0\ell0|s0)} H(\ell mLM) \ . \tag{7.50}$$

This shows explicitly how the $G^{(+)}$'s can be measured experimentally. Once they are measured, the spin and parity of the parent boson can be determined easily by using the formula (7.30).

Let us come back to the discussion of the $\pi\omega$ decay mode. The $G^{(+)}$'s in this case are given by relation (7.50) with $s = 1$. Explicitly, they are

$$G_{11}^{(+)}(LM) = \frac{1}{3}H(00LM) - \frac{5}{6}H(20LM) \qquad (\text{even } L) \tag{7.51a}$$

$$G_{10}^{(+)}(LM) = \frac{5}{2\sqrt{3}}H(21LM) \qquad (L \geq 1) \tag{7.51b}$$

$$G_{1-1}^{(+)}(LM) = -\frac{5}{\sqrt{6}}H(22LM) \qquad (L \geq 2) \tag{7.51c}$$

$$G_{00}^{(+)}(LM) = \frac{1}{3}H(00LM) + \frac{5}{3}H(20LM) \qquad (\text{ even } L) \ . \tag{7.51d}$$

The first information on the spin and parity can be obtained by noting the following. From Eq. (7.21a), we see that $g_{\lambda\lambda'}^J = 0$ for λ or $\lambda' = 0$, if $\varepsilon = \eta(-)^{J-1} = -1$. Therefore, if $\varepsilon = -1$,

$$G_{10}^{(+)}(LM) = G_{00}^{(+)}(LM) = 0 \ . \tag{7.52}$$

if this relation is not satisfied for some value of L and M, then we may conclude immediately that $\varepsilon = +1$.

If ε is known, we can determine the spin itself by dividing Eq. (7.51a) by Eq. (7.51c). Using formula (7.30), we obtain the following useful relation valid for $J \geq 1$:

$$\begin{aligned} \frac{G_{11}^{(+)}(LM)}{G_{1-1}^{(+)}(LM)} &= \varepsilon \frac{(J1L0|J1)}{(J-1L2|J1)} \\ &= -\varepsilon \left[\frac{(L-1)(L+2)}{L(L+1)}\right]^{\frac{1}{2}} \left[1 - \frac{L(L+1)}{2J(J+1)}\right] \qquad (\text{even } L \geq 2)\ , \end{aligned} \tag{7.53}$$

where one has used Eqs. (A.20) and (A.21). Of course, this formula has meaning only if the relevant $G^{(+)}$'s are non-zero for some value of L and M.

It is possible to obtain an additional spin formula applicable for the case $\varepsilon = +1$ and $J \geq 1$. If $\varepsilon = +1$, g_{00}^J is non-zero in general and related to g_{11}^J by

$$\frac{g_{11}^J}{g_{00}^J} = \frac{G_{11}^{(+)}(00)}{G_{00}^{(+)}(00)}\ . \tag{7.54}$$

Now, the desired spin formula can be obtained by taking the ratio of relations (7.51a) to (7.51d) and using Eq. (7.54):

$$\begin{aligned} \frac{G_{00}^{(+)}(00)}{G_{11}^{(+)}(00)} &= \frac{(J1L0|J1)}{(J0L0|J0)} \\ &= 1 - \frac{L(L+1)}{2J(J+1)} \qquad (\text{even } L \geq 2)\ . \end{aligned} \tag{7.55}$$

Therefore, if $\varepsilon = +1$ and $J \geq 1$, this formula supplies additional information on the spin J.

If $J = 0$, all H's should be zero for $L \geq 1$. In addition, we see that, from Eq. (7.30), the $G_{\lambda\lambda'}^{(+)}(00)$ vanish if λ or λ' is non-zero. Therefore, using Eq. (7.51a), we obtain $H(2000) = 2/5$. Note that $\eta = -1$ if $J = 0$, so that $\varepsilon = +1$.

7.4 $\pi_2(1670) \to \pi + f_2(1270)$

Let us consider, as a final example of our formalism, the problem of analyzing a boson resonance decaying into $\pi + f_2$. the f_2 meson is not a narrow resonance and the interference effect can be a serious problem, for our formalism does not apply in that case. However, the formalism can be applied, for instance, if the analysis is limited to the neutral decay mode $\pi^0 + f_2$ with f_2 decaying into the $\pi^+\pi^-$ system. In this case, there is no interference effect, because each of the three pions has a different charge.

As before, we start out by writing down the $G^{(+)}$'s explicitly (note $s^{\eta_s} = 2^+$):

$$G_{22}^{(+)}(LM) = \frac{1}{5}H(00LM) - H(20LM) + \frac{3}{10}(40LM) \qquad (\text{even } L) \tag{7.56a}$$

$$G_{21}^{(+)}(LM) = \sqrt{\frac{3}{2}}H(21LM) - \frac{3}{2\sqrt{5}}H(41LM) \qquad (L \geq 1) \tag{7.56b}$$

$$G_{20}^{(+)}(LM) \ = -H(22LM) + \frac{3}{2}\sqrt{\frac{3}{5}}H(42LM) \qquad (L \geq 2) \tag{7.56c}$$

$$G_{2-1}^{(+)}(LM) \ = -\frac{3}{2}\sqrt{\frac{7}{5}}H(43LM) \qquad (L \geq 3) \tag{7.56d}$$

$$G_{2-2}^{(+)}(LM) \ = 3\sqrt{\frac{7}{10}}H(44LM) \qquad (L \geq 4) \tag{7.56e}$$

$$G_{11}^{(+)}(LM) \ = \frac{1}{5}H(00LM) + \frac{1}{2}H(20LM) - \frac{6}{5}H(40LM) \qquad (\text{even } L) \tag{7.56f}$$

$$G_{10}^{(+)}(LM) \ = \frac{1}{2}H(21LM) + 3\sqrt{\frac{3}{10}}H(41LM) \qquad (L \geq 1) \tag{7.56g}$$

$$G_{1-1}^{(+)}(LM) \ = -\sqrt{\frac{3}{2}}H(22LM) - 3\sqrt{\frac{2}{5}}H(42LM) \qquad (L \geq 2) \tag{7.56h}$$

$$G_{00}^{(+)}(LM) \ = \frac{1}{5}H(00LM) + H(20LM) + \frac{9}{5}H(40LM) \qquad (\text{even } L)\ . \tag{7.56i}$$

We want to determine the value of $\varepsilon = \eta(-)^{J-1}$. If $\varepsilon = -1$, we have from Eq. (7.21a) that $g^J_{\lambda\lambda'} = 0$ for λ or $\lambda' = 0$. Thus, if $\varepsilon = -1$, we obtain the following conditions:

$$G_{00}^{(+)}(LM) \ = G_{10}^{(+)}(LM) = G_{20}^{(+)}(LM) = 0\ . \tag{7.57a}$$

Of course, this statement becomes non-trivial only for the allowed values of L as indicated in formulae (7.56a)–(7.56i). If $J = 1$ and $\varepsilon = -1$ (i.e. $H^\eta = 1^-$), both Eqs. (7.56b) and (7.56g) should be zero, so that one has the additional condition

$$H(21LM) \ = H(41LM) = 0 \qquad (L = 1 \text{ or } 2)\ . \tag{7.57b}$$

Only one parity state is allowed if $J = 0$, i.e. $\eta = -1$, so that $\varepsilon = +1$ in that case.

Once ε is known, we proceed to determine J in the following manner. First, consider the case $J = 0$. then, all the H's and $G^{(+)}$'s for $L \geq 1$ should vanish. In addition, $G^{(+)}_{\lambda\lambda'}(00) = 0$, if λ or $\lambda' \neq 0$, so that by using Eqs. (7.56a) and (7.56f), we obtain $H(2000) = H(4000 = 2/7$. Next, consider the case $J = 1$. Then, all the H's and $G^{(+)}$'s must vanish if $L \geq 3$. In addition, we have the condition $g^J_{\lambda\lambda'} = 0$ for λ or $\lambda' = 2$, so that

$$G_{22}^{(+)}(LM) = G_{21}^{(+)}(LM) = G_{20}^{(+)}(LM) = 0 \qquad (L = 0, 1, 2)\ . \tag{7.58}$$

Again, these relations are non-trivial only for those values of L as are indicated in Eqs. (7.56a), (7.56b), and (7.56c).

Now, we write down the spin-parity formula valid for all $J \geq 1$. Taking the ratio of Eqs. (7.56f) to (7.56h), we get

$$\frac{G_{11}^{(+)}(LM)}{G_{1-1}^{(+)}(LM)} = \varepsilon \frac{(J1L0|J1)}{(J-1L2|J1)} \qquad (\text{even } L \geq 2)\ , \tag{7.59}$$

where we have used the relation (7.30). Note that this formula is "formally" identical to formula (7.53). there exist three additional formulae, applicable if $J \geq 2$:

$$\frac{G_{21}^{(+)}(LM)}{G_{2-1}^{(+)}(LM)} \ = \ \varepsilon \frac{J1L1|J2)}{(J-1L3|J2)} \qquad (L \geq 3) \tag{7.60}$$

$$\frac{G^{(+)}_{22}(LM)}{G^{(+)}_{2-2}(LM)} = \varepsilon\frac{(J1L0|J2)}{(J-2L4|J0)} \qquad \text{(even } L \geq 4) \tag{7.61}$$

and

$$\frac{G^{(+)}_{11}(00)}{G^{(+)}_{22}(00)}\,\frac{G^{(+)}_{22}(LM)}{G^{(+)}_{11}(LM)} = \frac{(J2L0|J2)}{(J1L0|J1)} \qquad \text{(even } L \geq 2)\ . \tag{7.62}$$

Note that the formula (7.62) does not depend on the parity.

In analogy to the example of the B-meson decay, one can write down additional spin formulae, if $\varepsilon = +1$. We list them below for the sake of completeness. If $\varepsilon = +1$ and $J \geq 1$, one obtains

$$\frac{G^{(+)}_{00}(00)}{G^{(+)}_{11}(00)}\,\frac{G^{(+)}_{11}(LM)}{G^{(+)}_{00}(LM)} = \frac{(J1L0|J1)}{(J0L0|J0)} \qquad \text{(even } L \geq 2)\ . \tag{7.63}$$

If $\varepsilon = +1$ and $J \geq 2$, one has, in addition,

$$\frac{G^{(+)}_{00}(00)}{G^{(+)}_{22}(00)}\,\frac{G^{(+)}_{22}(LM)}{G^{(+)}_{00}(LM)} = \frac{(J2L0|J2)}{J0L0|J0)} \qquad \text{(even } L \geq 2)\ . \tag{7.64}$$

The ratios of Clebsch-Fordan coefficients appearing in Eqs. (7.59) to (7.64) may be expressed as explicit functions of J and L [see the formulae (A.20) and (A.24)].

8 Tensor Formalism for Integral Spin

In this and the next section, we consider the tensor wave functions describing relativistic particles with spin and satisfying the Rarita-Schwinger formalism [31]. Our main objective is to construct explicitly the tensor wave functions, following the method proposed by Auvil and Brehm [32], and show how they may be used to write down the covariant amplitudes for physical processes. The advantage of using the tensor formalism is that a spin tensor, its indices being those of four-momenta, can be coupled to any four-momenta and/or other spin tensors to form the simplest scalar amplitude satisfying the requirement of the Lorentz invariance.

As before, our main emphasis will be on the problem of describing resonance decays. We will consider in detail how the helicity amplitudes are related to the coupling constants appearing in the tensor formalism. In addition, we will demonstrate why it is possible to use the non-relativistic formalism of Zemach [33] in a relativistic problem. We shall see, in fact, that the non-relativistic formalism is more convenient to use in a purely phenomenological approach than the fully relativistic Rarita-Schwinger formalism.

In this section, we will concentrate on the problem of representing the integral spin states in the tensor formalism. Our first task is to construct the non-relativistic wave functions for particles of spin 1 known as the polarization vectors. They are the analogue of the state vectors $|jm\rangle$ with $j = 1$, but the Hilbert space of which they form the basis vectors is the familiar momentum space. As such, they carry indices of four-momenta as well as the magnetic quantum number m. We shall construct canonical as well as helicity state vectors and show that they transform in exactly the same way as the states $|jm\rangle$ or $|j\lambda\rangle$ under rotation.

The wave functions for spin 2 are given by the tensors of rank 2 which are constructed out of two polarization vectors by coupling them with the Clebsch-Gordan coefficients. The states of spin 2 have five independent components corresponding to the different values the magnetic quantum number can assume, whereas tensors of rank 2 with four-momentum indices have 16 independent components. This implies that a pure spin-2 tensor ought to satisfy subsidiary conditions which reduce the number of independent components to five. These subsidiary conditions are just those of the Rarita-Schwinger formalism. We shall see that the conditions for spin 2 can be easily generalized to those applicable to higher rank tensors describing particles of higher spin. Thensor wave functions corresponding to higher rank spin are constructed by coupling to the maximum possible spin, i.e. the rank of the tensors is equal to the spin. In this way, the tensor wave functions of arbitrary spin automatically satisfy the Rarita-Schwinger conditions.

8.1 Spin-1 states at rest

Consider an arbitrary three-momentum $\vec{p}$. It may be specified in terms of the three Cartesian orthonormal basis vectors $\vec{e}_i$ as follows:

$$\vec{p} = \sum_{i=1}^{3} p_i \, \vec{e}_i \ , \tag{8.1}$$

where $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are the unit vectors along the x-, y-, and z-axis, respectively. Alternatively, $\vec{p}$ can be expressed in terms of the spherical basis vectors $\vec{e}\,(m)$:

$$\vec{p} = \sum_{m} p(m) \, \vec{e}\,(m) \qquad (m = -1, 0, +1) \ , \tag{8.2}$$

where

$$\vec{e}\,(\pm 1) = \mp \frac{1}{\sqrt{2}} (\vec{e}_1 \pm i \vec{e}_2), \quad \vec{e}\,(0) = \vec{e}_3 \ . \tag{8.3}$$

The vectors $\vec{e}\,(m)$ are the polarization vectors in the three-momentum space; they correspond to spin-1 states at rest.

In order to gain a deeper understanding of these basis vectors, it is first necessary to consider the rotation matrices acting on the momentum $\vec{p}$ and obtain an explicit representation of the spin-1 angular momentum operators. For the purpose, we represent $\vec{e}_i$ as a column vector, so that $\vec{p}$ itself may be considered a column vector,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} . \tag{8.4}$$

Then, the rotation operator R acting on $\vec{p}$ is a 3×3 matrix:

$$R \, p_i = \sum_{j=1}^{3} R_{ij} \, p_j \qquad (i = 1, 2, 3) \ . \tag{8.5}$$

As before, we consider always the "active rotation", i.e. the rotation of momentum $\vec{p}$ with respect to a given *fixed* coordinate system.

Consider now rotations of $\vec{p}$ by ε with respect to the x-, y-, and z-axis to be denoted by $R_k(\varepsilon), k = 1, 2, 3$. They are exhibited explicitly below:

$$R_3(\varepsilon) = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon & 0 \\ \sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.6a}$$

$$R_2(\varepsilon) = \begin{pmatrix} \cos\varepsilon & 0 & \sin\varepsilon \\ 0 & 1 & 0 \\ -\sin\varepsilon & 0 & \cos\varepsilon \end{pmatrix} \tag{8.6b}$$

$$R_1(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} . \tag{8.6c}$$

Let us denote by S_k the infinitesimal generators of the rotations $R_k(\varepsilon)$:

$$R_k(\varepsilon) = e^{-i\varepsilon S_k} . \tag{8.7}$$

For formulae (8.6a)–(8.6c), we find that the matrices S_k have the following form (by considering the limit $\varepsilon \to 0$):

$$(S_k)_{\ell m} = -i\varepsilon_{k\ell m} . \tag{8.8}$$

The matrices S_k are Hermitian and satisfy the relations,

$$[S_k, S_\ell] = i\varepsilon_{k\ell m} S_m \tag{8.9a}$$

$$S^2 = S_k S_k = 2I \tag{8.9b}$$

where the summation is implied over repeated indices and I is a 3×3 unit matrix. These relations show immediately that the matrices S_k constitute a representation of angular momentum with eigenvalue one.

It is easy to see that the vectors $\vec{e}\,(m)$ given by Eqs. (8.3) are just the eigenvectors corresponding to angular momenta S_k:

$$\begin{aligned} S_3\, e(m) &= m\, e(m) \qquad (m = -1, 0, +1) \\ S_\pm\, e(\mp 1) &= \sqrt{2}\, e(0) \\ S_\pm\, e(\pm 1) &= 0 \\ S_\pm\, e(0) &= \sqrt{2}\, e(\pm 1) \end{aligned} \tag{8.10}$$

where $S_\pm = S_1 \pm iS_2$ and $\vec{e}\,(m)$ is the column vector given by

$$e(\pm 1) = \mp\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} , \quad e(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{8.3a}$$

Comparing Eq. (8.10) with Eq. (1.2), we see that the vectors $\vec{e}\,(m)$ represent the spin-1 wave functions, similar to the state vectors $|jm\rangle$ ($j = 1$) discussed in Section 1. As such, they carry two different sets of indices, one corresponding to the z-component of angular momentum and the other corresponding to the three-momentum index. It is interesting to note that, once the angular momentum operator is given by Eq. (8.8), the corresponding eigenvectors satisfying Eqs. (8.10) [or more generally Eqs. (1.2)] can be deduced by considering S_3 and $S_\pm$; it can be shown that the simplest solution is just the set of eigenvectors given in formulae (8.3a).

From Eq. (1.5), one finds that under a rotation $R = R(\alpha,\beta,\gamma)$, the vector $\vec{e}\,(m)$ transforms according to

$$R(\alpha,\beta,\gamma)\, e_i(m) = \sum_{m'} D^{(1)}_{m'm}(\alpha,\beta,\gamma)\, e_i(m') \qquad (i = 1,2,3)\ . \tag{8.11}$$

We have dropped the symbol $U[\ ]$ from the operator representing a rotation $R(\alpha,\beta,\gamma)$, in order to emphasize the fact that canonical states $\vec{e}\,(m)$ are *not* the basis vectors of some abstract Hilbert space but those of the familiar three-momentum space. There is an alternative way of representing a rotation on $\vec{e}\,(m)$. It follows from the fact that both the momentum $\vec{p}$ and the polarization vector $\vec{e}\,(m)$ may be expanded in terms of the same set of basis vectors $\vec{e}_i$ [see Eqs. (8.1) and (8.3)]. If R_{ij} is a 3×3 matrix acting on the components p_i as shown in Eq. (8.5), one finds

$$R(\alpha,\beta,\gamma)\, e_i(m) = \sum_{j} R_{ij}(\alpha,\beta,\gamma)\, e_j(m) \qquad (m = -1,0,+1)\ . \tag{8.12}$$

The rotation matrix $R(\alpha,\beta,\gamma)$ may be expressed in terms of the matrices R_k of Eqs. (8.6a)–(8.6c) as follows:

$$R(\alpha,\beta,\gamma) = R_3(\alpha)R_2(\beta)R_3(\gamma)\ . \tag{8.13}$$

The reader may easily check that Eqs. (8.11) and (8.12) are indeed identical by using the explicit expressions for $D^{(1)}_{m'm}$ and Eq. (8.13).

The relation (8.12) may be used to show that the inner product of $\vec{e}\,(m)$ with an arbitrary momentum vector is rotationally invariant, i.e.

$$\vec{p}\cdot\vec{e}\,(m) = \vec{p}\,'\cdot\vec{e}\,'(m)\ , \tag{8.14}$$

where

$$p_i' = R_{ij}p_j \quad \text{and} \quad e_i'(m) = R_{ij}\, e_j(m)\ . \tag{8.15}$$

This follows, of course, from the fact that the rotation matrices are orthogonal:

$$R^{-1}_{ij} = \tilde{R}_{ij} = R_{ji}\ . \tag{8.16}$$

In other words, the inverse of a rotation matrix is equal to its transpose. As we shall see later, it is the property (8.14) which enables one to mix $\vec{p}$ and $\vec{e}\,(m)$ to construct rotationally invariant amplitudes.

Finally, let us note the following properties of the polarization vectors:

$$\vec{e}^{\,*}(m)\cdot\vec{e}\,(m') = \delta_{mm'} \tag{8.17a}$$

$$\sum_m e_i\,(m)\, e_j^*(m) = \delta_{ij} \tag{8.17b}$$

$$\vec{e}^{\,*}(m) = (-)^m\, \vec{e}\,(-m)\ . \tag{8.17c}$$

8.2 Relativistic spin-1 wave functions

The polarization vectors $\vec{e}(m)$ we have considered so far are, of course, the canonical state vectors (or wave functions) describing particles of spin 1 at rest. By definition, spin characterizes how a particle at rest behaves under spatial rotations. It follows, therefore, that it cannot have the energy component, if the spin wave function is to be represented in the momentum space. With this in mind, we define a four-vector describing a spin-1 particle at rest,

$$\begin{aligned} e^\mu(0,m) &= \{0, \vec{e}(m)\} \\ e_\mu(0,m) &= \{0, -\vec{e}(m)\}\ . \end{aligned} \tag{8.18}$$

We are now ready to define the canonical and helicity state vectors. In analogy to Eqs. (2.14) and (2.16), we write

$$e^\mu(\vec{p},m) = [\overset{\circ}{R} L_z(p) \overset{\circ}{R}{}^{-1}]^\mu{}_\nu\ e^\nu(0,m) \tag{8.19}$$

and

$$e^\mu(\vec{p},\lambda) = (\overset{\circ}{R} L_z(p)]^\mu{}_\nu\ e^\nu(0,\lambda)\ , \tag{8.20}$$

where $L_z(p)$ is given in Eq. (2.6) and $\overset{\circ}{R}$ is the rotation which takes the z-axis (or $\vec{e}_3$) into the direction of $\vec{p}$, i.e. $\hat{p} = (\theta, \phi)$,

$$\overset{\circ}{R}{}^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & \overset{\circ}{R}_{ij} \end{pmatrix}$$

and $\overset{\circ}{R}_{ij}$ is the 3×3 rotation matrix $R(\phi, \theta, 0)$ of Eq. (8.13). If the momentum $\vec{p}$ is along the z-axis, both Eqs. (8.19) and (8.20) have the same form. Explicitly, one obtains

$$\begin{aligned} e^\mu(p\hat{z}, \pm 1) &= \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) \\ e^\mu(p\hat{z}, 0) &= \left(\frac{p}{w}, 0, 0, \frac{E}{w}\right)\ , \end{aligned} \tag{8.21}$$

where E, P, and w are the energy, momentum, and mass of the spin-1 particle.

The property (8.11) guarantees that the states given by Eqs. (8.19) and (8.20) transform under rotation in the same way as the ket vectors $|\vec{p}, jm\rangle$ and $|\vec{p}, j\lambda\rangle$ discussed in Section 2. Therefore, we have

$$R\, e^\mu(\vec{p},m) = \sum_{m'} D^{(1)}_{m'm}(R)\ e^\mu(R\vec{p}, m') \tag{8.22}$$

and

$$R\, e^\mu(\vec{p},\lambda) = e^\mu(R\vec{p},\lambda)\ . \tag{8.23}$$

As before, the helicity vectors are related to the canonical vectors via

$$e^\mu(\vec{p},\lambda) = \sum_m D^{(1)}_{m\lambda}(\overset{\circ}{R})\ e^\mu(\vec{p},m)\ , \tag{8.24}$$

where $\overset{\circ}{R}$ is the rotation that appears in Eq. (8.20).

Listed below are a number of properties satisfied by the polarization four-vectors:

$$p^\mu e_\mu(\vec{p}, m) \ = 0 \tag{8.25a}$$
$$e^*_\mu(\vec{p}, m)\, e^\mu(\vec{p}, m') \ = -\delta_{mm'} \tag{8.25b}$$
$$P^{(1)}_{\mu\nu} = \sum_m e_\mu(\vec{p}, m)\ e^*_\nu(\vec{p}, m) = \bar{g}_{\mu\nu}(p) \tag{8.26a}$$
$$\bar{g}_{\mu\nu}(p) = -g_{\mu\nu} \ + \frac{p_\mu p_\nu}{w^2}\ , \tag{8.26b}$$

where $P^{(1)}_{\mu\nu}$ is the spin-1 projection operator to be discussed below and $\bar{g}_{\mu\nu}$ is an object which reduces to δ_{ij} (no energy component) in the rest system of the particle with mass w. The relation (8.25a) is a consequence of our definition (8.18); it may be viewed as a necessary condition to be satisfied by a spin-1 wave function (with three independent components), when it has been imbedded into a four-dimensional space. The relations (8.25b), (8.26a) and (8.26b) are generalizations of the rest-state formulae (8.17a) and (8.17b). Note that the tensors $P^{(1)}_{\mu\nu}$ satisfy

$$P^{(1)}_{\mu\alpha}\, P^{(1)\,\alpha}{}_\nu = -P^{(1)}_{\mu\nu}\ . \tag{8.27}$$

Therefore, $P^{(1)}_{\mu\nu}$ may be considered as a projection operator in the sense that, when it is applied to any four-vector, the resulting four-vector is orthogonal to p_μ owing to the property (8.25a). We point out that there exist relations identical to (8.25a), (8.25b), (8.26a), and (8.26b) for the helicity state vectors $e^\mu(\vec{p}, \lambda)$.

Let us briefly discuss the parity and time-reversal operations on the polarization vectors. First, the rest-states transform as follows:

$$\boldsymbol{P}\, \vec{e}\,(m) = \eta\, \vec{e}\,(m) \tag{8.28}$$

$$\boldsymbol{T}\, \vec{e}\,(m) = (-)^{1-m}\, \vec{e}\,(-m) = -\vec{e}^{\,*}(m)\ , \tag{8.29}$$

where one has used Eqs. (3.7) and (3.8). The polarization four-vectors transform under $\boldsymbol{P}$ and $\boldsymbol{T}$ according to Eqs. (3.9) and (3.10):

$$\boldsymbol{P}\, e_\mu(\vec{p}, m) \ = \eta\, e_\mu(-\vec{p}, m) = -\eta \boldsymbol{P}_\mu{}^\nu\ e_\nu(\vec{p}, m) \tag{8.30}$$
$$\boldsymbol{T}\, e_\mu(\vec{p}, m) \ = (-)^{1-m}\, e_\mu(-\vec{p}, -m) = \boldsymbol{P}_\mu{}^\nu\ e^*_\nu(\vec{p}, m)\ , \tag{8.31}$$

where $\boldsymbol{P}_\mu{}^\nu$ denotes the space inversion, i.e.

$$\boldsymbol{P}_\mu{}^\nu = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}\ . \tag{8.32}$$

According to Eqs. (3.11) and (3.12), the helicity vectors transform as follows:

$$\boldsymbol{P}\, e_\mu(\vec{p}, \lambda) \ = -\eta\, e_\mu(-\vec{p}, -\lambda) = \eta\, \boldsymbol{P}_\mu{}^\nu\ e_\nu(\vec{p}, -\lambda) \tag{8.30a}$$
$$\boldsymbol{T}\, e_\mu(\vec{p}, \lambda) \ = (-)^\lambda\, e_\mu(\vec{p}, \lambda) = -\boldsymbol{P}_\mu{}^\nu\ e^*_\nu(\vec{p}, -\lambda)\ . \tag{8.31a}$$

8.3 Spin-2 and higher-spin wave functions

The wave functions describing spin-2 particles can be constructed out of the polarization vectors as follows:

$$e_{\mu\nu}(\vec{p}, 2m) = \sum_{m_1 m_2} (1m_1 1m_2|2m)\, e_\mu(\vec{p}, m_1)\, e_\nu(\vec{p}, m_2)\ . \tag{8.33}$$

The rotational property (8.22) for the polarization vectors guarantees that our spin-2 states have the correct rotational property:

$$R\, e_{\mu\nu}(\vec{p}, 2m) = \sum_{m'} D^{(2)}_{m'm}(R)\ e_{\mu\nu}(R\vec{p}, 2m')\ . \tag{8.34}$$

The spin-2 helicity states are given in terms of the spin-1 helicity vectors in exactly the same way as the spin-2 canonical states in Eq. (8.33); one merely replaces the m's in Eq. (8.33) by the λ's. The relation (8.24) can be used to show that the spin-2 helicity states are correctly related to the canonical states:

$$e_{\mu\nu}(\vec{p}, 2\lambda) = \sum_{m} D^{(2)}_{m\lambda}(\mathring{R})\ e_{\mu\nu}(\vec{p}, 2m)\ , \tag{8.35}$$

where, as before, $\mathring{R}$ describes the $\vec{p}$ direction.

Now, spin-2 states have five independent components corresponding to the number of different values the z-component of spin can take. The formula (8.33) shows, on the other hand, that our spin-2 wave function is a tensor of rank 2 with sixteen independent components. This implies that there exist supplementary conditions which reduce the number of independent components to 5. These are just the Rarita-Schwinger conditions for the integral-spin tensors. From the definition (8.33), we can show, in fact, that

$$p^\mu\, e_{\mu\nu}(\vec{p}, 2m) = 0 \tag{8.36a}$$

$$e_{\mu\nu}(\vec{p}, 2m) = e_{\nu\mu}(\vec{p}, 2m) \tag{8.36b}$$

$$g^{\mu\nu}\, e_{\mu\nu}(p, 2m) = 0\ . \tag{8.36c}$$

It is easy to see that these conditions limit to five the number of independent components in $e_{\mu\nu}$.

In the rest frame of the spin-2 particle, the relations (8.36) reduce to

$$e_{ij}(2m) = e_{ji}(2m) \tag{8.37a}$$

$$\sum_i e_{ii}(2m) = 0\ . \tag{8.37b}$$

The condition (8.36a) simply ensures that in the rest frame indices μ and ν can have the space components only, i.e. 1, 2, or 3. Equations (8.37) tell us that the tensors e_{ij} are symmetric and traceless.

Let us note that $e_{\mu\nu}$ is normalized according to

$$e^*_{\mu\nu}(\vec{p}, 2m)\ e^{\mu\nu}(\vec{p}, 2m') = \delta_{mm'}\ , \tag{8.38}$$

which can be shown using Eqs. (8.33) and (8.25b). In analogy to Eq. (8.26a), we may also define a spin-2 projection operator:

$$P^{(2)}_{\mu\nu\alpha\beta} = \sum_m e_{\mu\nu}(\vec{p}, 2m)\, e^*_{\alpha\beta}(\vec{p}, 2m) \tag{8.39}$$

with the normalization given by

$$P^{(2)}_{\mu\nu\sigma\rho}\, P^{(2)\,\sigma\rho}{}_{\alpha\beta} = P^{(2)}_{\mu\nu\alpha\beta}\ . \tag{8.40}$$

Again, the spin-2 projection operator has the property that, when it is applied to any second-rank tensor, the resulting tensor satisfies all the conditions in Eqs. (8.36). In the rest frame, this operator simply projects out that part of a second-rank tensor which is symmetric and traceless. From this, one sees immediately that in the rest frame

$$P^{(2)}_{ijk\ell} = \frac{1}{2}\,(\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk}) - \frac{1}{3}\,\delta_{ij}\,\delta_{k\ell}\ . \tag{8.41}$$

A projection operator defined in an arbitrary frame out to reduce to formula (8.41) in the rest frame, so that it can be written

$$P^{(2)}_{\mu\nu\alpha\beta} = \frac{1}{2}\,(\bar{g}_{\mu\alpha}\,\bar{g}_{\nu\beta} + \bar{g}_{\mu\beta}\,\bar{g}_{\nu\alpha}) - \frac{1}{3}\,\bar{g}_{\mu\nu}\,\bar{g}_{\alpha\beta}\ , \tag{8.42}$$

where $\bar{g}_{\mu\nu}$ is given in Eq. (8.26b) and reduces to δ_{ij} in the rest frame.

It is clear how one can generalize to high spin states the results we have developed so far for spin-2 states. Let us briefly discuss the case of spin-3 states. They may be described in general by a third-rank tensor

$$e_{\mu\nu\sigma}(\vec{p}, 3m) = \sum_{m_1 m_2} (2m_1 1m_2|3m)\, e_{\mu\nu}(\vec{p}, 2m_1)\, e_\sigma(\vec{p}, m_2) \tag{8.43}$$

with the normalization,

$$e^*_{\mu\nu\gamma}(\vec{p}, 3m)\, e^{\mu\nu\sigma}(\vec{p}, 3m') = -\delta_{mm'}\ . \tag{8.43a}$$

Proper rotational property is guaranteed by construction, implying that it is indeed a wave function appropriate for a spin-3 state. Therefore, the wave functions should satisfy the Rarita-Schwinger conditions:

$$p^\mu\, e_{\mu\nu\sigma} = 0 \tag{8.44a}$$

$$e_{\mu\nu\sigma} = \text{pairwise symmetric} \tag{8.44b}$$

$$g^{\mu\nu}\, e_{\mu\nu\gamma} = 0\ . \tag{8.44c}$$

The condition (8.44b) states that $e_{\mu\nu\sigma}$ should remain invariant under interchange of any two indices. In the rest frame, $e_{\mu\nu\sigma}$ reduces to e_{ijk} with only space indices and it is symmetric (pair-wise) and traceless.

The spin-3 projection operators can be constructed in a similar fashion as in Eq. (8.39):

$$P^{(3)}_{\mu\nu\sigma\alpha\beta\gamma} = \sum_m e_{\mu\nu\gamma}(\vec{p}, 3m)\, e^*_{\alpha\beta\gamma}(\vec{p}, 3m)\ . \tag{8.45}$$

In the rest frame, this operator projects out a symmetric and traceless tensor from an arbitrary third-rank tensor. Using this fact, it is easy to construct a projection operator in the rest frame analogous to formula (8.41):

$$P^S_{ijk\ell mn} = \frac{1}{6} \sum_p \delta_{i\ell}\, \delta_{jm}\, \delta_{kn} \tag{8.45a}$$

$$P^{(3)}_{ijk\ell mn} = P^S_{ijk\ell mn} - \frac{1}{5} \left[\delta_{ij}\, P^S_{aak\ell mn} + \delta_{jk}\, P^S_{aai\ell mn} + \delta_{ki}\, P^S_{aaj\ell mn}\right] , \tag{8.45b}$$

where the summation in the first relation goes over the six possible permutations of the three indices (i, j, k). The second relation is clearly traceless in any pair of the indices (i, j, k). The formulae (8.45b) and (8.41) are two special cases of the general formula given by Zemach [33]. In order to obtain a projection operator valid in an arbitrary frame, one merely needs to replace the Kronecker δ's in Eq. (8.45a) by the $\bar{g}$'s of Eq. (8.26b). The reader is referred to Fronsdal [34] for an explicit expression of relativistic projection operators of arbitrary spin.

8.4 Zemach formalism

Let us now turn to a discussion of how the projection operators may be used to facilitate calculations in actual problems. For the purpose, it is best to consider an example; let us suppose that we wish to describe the production and decay of the $f_2(1270)$ meson. Then, the simplest Lorentz-invariant amplitude will assume the form

$$\mathcal{M}_{fi} \sim \sum_m p_1^\mu\, p_2^\nu\, e_{\mu\nu}(\vec{p}, 2m)\, e^*_{\alpha\beta}(\vec{p}, 2m)\, q_1^\alpha\, q_2^\beta , \tag{8.46}$$

where $\vec{p}$ is the momentum of the f meson, p_1 and p_2 are the decay pion momenta from the f meson, and q_1 and q_2 are some momenta taken from the production process (the Breit-Wigner function for the f propagator has been suppressed).

Let us explain at this point the meaning of the amplitude (8.46). The isospin-zero $\pi\pi$ scattering at the f mass is really a product of two processes, i.e. the production and decay of the f meson (see Fig. 2). The amplitude corresponding to each process should be Lorentz invariant and exhibit the degree of freedom corresponding to the z-component of the spin (the quantum number m). Furthermore, one has to sum over the quantum numbers m in the amplitude, since the f meson is an intermediate resonance and is not observed directly. Note that all these requirements are neatly satisfied by the expression (8.46). The complex conjugation of the second wave function in the expression signifies the production (and not the decay) of a resonance.

Using the definition (8.39), $\mathcal{M}_{fi}$ may be written in terms of the projection operator:

$$\mathcal{M}_{fi} \sim p_1^\mu p_2^\nu\, P^{(2)}_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta . \tag{8.46a}$$

This example shows that is not in general necessary to construct explicitly the tensor wave functions; one only needs to know the corresponding projection operators. In the f rest frame, the amplitude can be expressed as follows:

$$\mathcal{M}_{fi} \sim p_1^i p_2^j\, T^{(2)}_{ij}(q_1 q_2) , \tag{8.47}$$

where $T_{ij}^{(2)}$ is a symmetric and traceless tensor given by

$$T_{ij}^{(2)}(q_1q_2) = P_{ijk\ell}^{(2)}\, q_1^k q_2^\ell \,. \tag{8.48}$$

Eq. (8.46a) may be reduced to the following equivalent form:

$$\mathcal{M}_{fi} \sim q_1^i q_2^j\, T_{ij}^{(2)}(p_1p_2) \,, \tag{8.49}$$

where

$$T_{ij}^{(2)}(p_1p_2) = P_{ijk\ell}^{(2)}\, p_1^k p_2^\ell \,. \tag{8.50}$$

Both Eqs. (8.47) and (8.49) tell us how to construct amplitudes: out of the decay (or production) momenta, construct a "pure" tensor (symmetric and traceless) and combine it with a "raw" tensor built out of the production (or decay) momenta, to obtain the desired amplitude.

This example illustrates the use of angular momentum tensors proposed by Zemach [33] in its simplest form. In his formalism, the pure spin tensors play the central role, thereby avoiding the explicit construction of spin wave functions. In addition, by evaluating amplitudes always in the rest frame of the particle to be described, his formalism avoids the complication arising from the use of the four-momentum indices. Listed below are a few of the pure spin tensors (i.e. symmetric and traceless):

$$T_i^{(1)}(a) = a_i \tag{8.51}$$

$$T_{ij}^{(2)}(ab) = \frac{1}{2}(a_ib_j + b_ia_j) - \frac{1}{3}\delta_{ij}(\vec{a}\cdot\vec{b}\,) \tag{8.52}$$

$$T_{ijk}^{(3)}(a) = a_ia_ja_k - \frac{a^2}{5}(\delta_{ij}a_k + \delta_{jk}a_i + \delta_{ki}a_j) \,, \tag{8.53}$$

where $\vec{a}$ and $\vec{b}$ are arbitrary three-vectors. A general prescription for constructing pure tensors of arbitrary spin can be found in Zemach [33].

The Zemach formalism is particularly suited to problems where interference effects occur due to the presence of identical particles. Let us discuss this problem by analyzing the decay of $a_2(1320)$ as an example. For simplicity, we shall discuss the decay of a positively charged a_2:

$$a_2^+(p) \to \pi^+(p_1) + \rho^0(p_{23}), \quad \rho^0(p_{23}) \to \pi^+(p_2) + \pi^-(p_3) \tag{8.54a}$$

$$a_2^+(p) \to \pi^+(p_2) + \rho^0(p_{13}), \quad \rho^0(p_{13}) \to \pi^+(p_1) + \pi^-(p_3). \tag{8.54b}$$

where we have indicated the momenta of the particles in parenthesis. Let us denote by w and w_{23} (or w_{13}) the invariant masses of the a_2 and ρ^0, respectively.

In a Dalitz plot analysis of the 3π final state, one may integrate over the orientation of the 3π system. From Eq. (6.20), one sees that in this case the distribution is independent of the density matrix of the a_2 (i.e. independent of the production process):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}w\mathrm{d}w_{13}^2\mathrm{d}w_{23}^2} \sim \sum_m |F_m|^2 \,, \tag{8.55}$$

where m is the z component of the a_2 spin along the normal to the 3π plane. The decay amplitude F_m may be expressed in terms of the $a_2(J^p = 2^+)$ wave function $e_{\mu\nu}$:

$$F_m \sim \phi^{\mu\nu} e_{\mu\nu}(\vec{p}, 2m) , \tag{8.56}$$

where $\phi^{\mu\nu}$ is a second-rank tensor to be built out of the variables of the 3π system. Then,

$$\begin{aligned} \sum_m |F_m|^2 &\sim \phi^{\mu\nu} e_{\mu\nu} e^*_{\alpha\beta} \phi^{*\alpha\beta} \\ &\sim \phi^{\mu\nu} P^{(2)}_{\mu\nu\alpha\beta} \phi^{*\alpha\beta} \\ &\sim \phi^{*\mu\nu} P^{(2)}_{\mu\nu\alpha\beta} \phi^{\alpha\beta} , \end{aligned} \tag{8.57}$$

where the last line has been obtained using the reality of the projection operators. In the a_2 rest frame, formula (8.57) becomes

$$\sum_m |F_m|^2 \sim \phi^{*ij} \; P^{(2)}_{ijk\ell} \; \phi^{k\ell} . \tag{8.57a}$$

Let us now construct explicitly the tensor $\phi^{\mu\nu}$. Taking into account the Bose symmetrization between the two π^+'s, the tensor can be written

$$\phi^{\mu\nu} = D(w_{23}) \, \phi^{\mu\nu}_{23} + D(w_{13}) \, \phi^{\mu\nu}_{13} , \tag{8.58}$$

where the first term corresponds to the process (8.54a) and the second term to the process (8.54b). $D(w)$ is the usual Breit-Wigner function with invariant mass w. $\phi^{\mu\nu}_{23}$ describes the case in which the intermediate ρ^0 is formed out of $\pi^+(p_2)$ and $\pi^-(p_3)$:

$$\phi^{\mu\nu}_{23} = \sum_{m_1} (p_2 - p_3)^\sigma \; e_\sigma(\vec{p}_{23}, m_1) \; e^*_\alpha(\vec{p}_{23}, m_1) \; p_{1\beta} \, p_\gamma \; \varepsilon^{\mu\alpha\beta\gamma} \; p^\nu_1 . \tag{8.59}$$

It can be shown that the decay amplitude given by

$$\phi^{\mu\nu}_{23} \; e_{\mu\nu}(\vec{p}, 2m) \tag{8.60}$$

is invariant under parity operation by using Eqs. (8.30) and (8.33) [the presence of the totally antisymmetric 4th rank tensor in Eq. (8.59) can be understood in this way]. Using the spin-1 projection operator Eq.(8.26a), the tensor $\phi^{\mu\nu}_{23}$ can be rewritten

$$\phi^{\mu\nu}_{23} = (p_2 - p_3)_\alpha \; p_{1\beta} \, p_\gamma \, \varepsilon^{\mu\alpha\beta\gamma} \; p^\nu_1 . \tag{8.61}$$

In the a_2 rest frame, the four-vector p has only the energy component, so that Eq. (8.61) becomes

$$\phi^{ij}_{23} \sim q^i_{23} \, p^j_1, \quad \vec{q}_{23} = (\vec{p}_2 - \vec{p}_3) \times \vec{p}_1 . \tag{8.62}$$

Similarly,

$$\phi^{ij}_{13} \sim q^i_{13} \, p^j_2, \quad \vec{q}_{13} = (\vec{p}_1 - \vec{p}_3) \times \vec{p}_2 . \tag{8.63}$$

Let us write down explicitly the factors appearing in Eq. (8.57a):

$$\phi^{*\,ij} = D^*(w_{23}) \, \phi^{ij}_{13} + D^*(w_{13}) \, \phi^{ij}_{13} \tag{8.63a}$$

$$P^{(2)}_{ijk\ell} \, \phi^{k\ell} = D(w_{23}) \, T^{(2)}_{ij}(q_{23}, p_1) + D(w_{13}) \, T^{(2)}_{ij}(q_{13}, p_2) . \tag{8.63b}$$

$T^{(2)}_{ij}$ is the pure spin-2 tensor given in (8.52).

This example illustrates how one may construct the general distribution function for the 3π Dalitz plot. The prescription is this: for any assumed spin-parity of the a_2, construct a raw tensor out of the variables in the 3π system consistent with the spin-parity and the Bose symmetrization [e.g. Eq. (8.63a)], form a pure tensor through the use of the projection operators, and multiply the two tensors together as shown in Eq. (8.57a), to obtain the desired distribution function. This demonstrates the convenience of employing the Zemach formalism for this type of problem.

8.5 Decay amplitudes in tensor formalism

Through a series of simple examples, we wish to show next how to construct invariant amplitudes for resonance decays in the tensor formalism. The purpose is two-fold. Firstly, we wish to demonstrate the connection between the decay amplitudes given in the tensor formalism to those derived in the helicity formalism. Secondly, we wish to show how one can use in a phenomenological approach the non-relativistic rest-state wave function in relativistic problems.

The examples we shall consider are all special cases of the following general problem, i.e. decay of a resonance J into a particle s by a pion emission:

$$J^{\eta}(p) \to s^{\eta_s}(p_1) + \pi(p_2) \ , \tag{8.64}$$

where the spin-parity and momentum for each particle are indicated in an obvious way. Let us denote by w and w_1 the invariant mass of J and s, respectively. Let $\omega = (\theta, \phi)$ be the spherical angles of $\vec{p}_1$ in the rest frame of $J(J\mathrm{RF})$. Then, the decay amplitude is given, from Eq. (5.16), by

$$A = N_J F^J_\lambda D^{J\,*}_{m\lambda}(\phi, \theta, 0) \tag{8.65}$$

and, from parity conservation,

$$F^J_\lambda = \varepsilon F^J_{-\lambda}, \quad \varepsilon = \eta\eta_s(-)^{J-s+1} \ . \tag{8.66}$$

The helicity amplitude has the following partial-wave expansion [from Eq. (5.18)]:

$$F^J_\lambda = \sum_\ell \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} a^J_\ell \ (\ell 0 s\lambda | J\lambda) \ . \tag{8.67}$$

The parity conservation implies that the partial-waves ℓ can take only even (odd) values, if the factor $\eta\eta_s$ is odd (even).

Let us now see how the decay amplitudes constructed in the tensor formalism may be cast into forms similar to Eq. (8.65). Wherever possible, we shall use the symbols introduced in the previous paragraph.

i) $\underline{2^+ \to 0^- + 0^-}$

The invariant amplitude for this decay may be written

$$\begin{aligned} A &\sim g\, p_1^\mu\, p_2^\nu\; e_{\mu\nu}(\vec{p}, 2m) \\ &\sim g\, p_1^i\, p_2^j\; e_{ij}(2m) \ , \end{aligned} \tag{8.68}$$

where g is the coupling constant and the second line has been evaluated in the JRF. Let us rotate the wave function e_{ij} by $R(\phi, \theta, 0)$:

$$e'_{ij}(2m) = R\, e_{ij}(2m) = \sum_{m'} D^{(2)}_{m'm}(\phi, \theta, 0) e_{ij}(2m') \ .$$

This can be inverted to obtain

$$e_{ij}(2m) = \sum_{m'} D^{(2)\,*}_{mm'}(\phi, \theta, 0)\, e'_{ij}(2m') \ . \tag{8.69}$$

e'_{ij} is by definition a wave function with the quantization axis along the direction $\vec{p}_1$. Using this fact, one can easily evaluate the following term:

$$p_1^i\, p_2^j\, e'_{ij}(2m') = \sum_{m_1 m_2} (1m_1 1m_2|2m')\, p_1^i\, e'_1(m_1)\, p_1^j\, e'_j(m_2) = p^2\, (1010|20)\, \delta_{m',0} \ , \tag{8.70}$$

where one has used the relation

$$\vec{p}_1 \cdot \vec{e}\,'(m) = p_1 \delta_{m,0} \ . \tag{8.71}$$

This relation follows from the fact that the quantization axis coincides with the direction of $\vec{p}_1$ and that the momentum $\vec{p}_1$ may be thought of as having only a z-component.

Combining Eqs. (8.69) and (8.70), one can recast the amplitude A in Eq. (8.68) into

$$A \sim \sqrt{\frac{2}{3}}\, g\, p_1^2\, D^{(2)\,*}_{m0}(\phi, \theta, 0) \ . \tag{8.72}$$

Of course, one could have written down this formula immediately by applying Eq. (8.65). The formula (8.72), however, tells us something further; it indicates that the helicity amplitude F ought to have the dependence p_1^2, commensurate with the D-wave in the di-pion system. We shall see that this is a general feature with the tensor formalism. Namely, the amplitude F^J_λ, which is an undefined quantity in the helicity formalism except for the symmetry property (8.66), has a more explicit representation in the tensor formalism. This point is illustrated further in the next example.

ii) $2^+ \rightarrow 1^- + 0^-$

This decay process has been considered previously. Let us write the amplitude in the following way:

$$\begin{aligned} A &\sim g\, \varepsilon^{\mu\alpha\beta\gamma}\, e^*_\alpha(\vec{p}_1, \lambda)\, p_{2\beta}\, p_\gamma\, p_2^\nu\, e_{\mu\nu}(\vec{p}, 2m) \\ &\sim g\, [\vec{e}^{\,*}(\vec{p}_1, \lambda) \times \vec{p}_2]^i\, p_2^j\, e_{ij}(2m) \ , \end{aligned} \tag{8.73}$$

where λ is the helicity of the 1^- particle. As before, we express e_{ij} in terms of the e'_{ij} whose quantization axis is along the $\vec{p}_1$ direction:

$$\begin{aligned} e_{ij}(2m) &= \sum_{m'} D^{(2)\,*}_{mm'}(\phi, \theta, 0)\, e_{ij}(2m') \\ &= \sum_{\substack{m' \\ m_1 m_2}} D^{(2)\,*}_{mm'}(\phi, \theta, 0)\, (1m_1 1m_2|2m')\, e'_i(m_1)\, e'_j(m_2) \ . \end{aligned} \tag{8.74}$$

Note the following relation,

$$[\vec{e}^{\,*}(p_1,\lambda)\times\vec{p}_2]^i\;e'_i(m_1)=i\,p_1\,\lambda\,\delta_{\lambda,m_1}\;. \tag{8.75}$$

which may be obtained by using expression (8.21). Combining this with Eq. (8.71), one obtains finally

$$A\sim g\,p_1^2\,\lambda(101\lambda|2\lambda)\;D^{(2)\,*}_{m\lambda}(\phi,\theta,0)\;. \tag{8.76}$$

This exercise shows that the helicity amplitude F^J_λ has the following explicit expression

$$F^{(2)}_\lambda\sim g\,p_1^2(101\lambda|2\lambda)\;. \tag{8.77}$$

Note that $F^{(2)}_\lambda$ satisfies the symmetry relations of Eqs. (8.66). Let us compare expression (8.77) with the amplitude obtained by using Eq. (8.67) with $\ell=2$:

$$F^{(2)}_\lambda\sim a_2\,(201\lambda|2\lambda)\;. \tag{8.77a}$$

We see that formula (8.77a) is equivalent to formula (8.77), if we set $a_2\sim gp_1^2$.

Next, let us consider the following non-relativistic amplitude:

$$A\sim g\,[\vec{e}^{\,*}(\lambda)\times\vec{p}_2]^i\;p_2^j\;e_{ij}(2m)\;, \tag{8.78}$$

where $\vec{e}\,(\lambda)$ is the polarization vector evaluated in the sRF (or the 1^- particle rest frame). Formula (8.78) is clearly non-relativistic, because $\vec{e}\,(\lambda)$ and $e_{ij}(2m)$ are the wave functions evaluated in two different rest frames. However, it can be shown easily that Eq. (8.78) leads to the same result (8.76). In this sense, Eq. (8.78) is an equally valid description of the decay process as the fully relativistic amplitude (8.73). We have now seen the simplest example which demonstrates how it is possible to use non-relativistic wave functions in a relativistic problem. A more complicated case involving two different amplitudes is given in the next example.

iii) $\underline{2^-\to1^-+0^-}$

This decay involves in general two independent amplitudes, reflecting the fact that there can be two orbital angular momenta, i.e. P- and F-waves. Let us write the invariant amplitude as follows:

$$A\sim g_1\,e^{\mu\,*}(\vec{p}_1,\lambda)\;p_1^\nu\;e_{\mu\nu}(\vec{p},2m)+g_3\,e^{\alpha\,*}(\vec{p}_1,\lambda)\;p_\alpha\,p_1^\mu\,p_1^\nu\;e_{\mu\nu}(\vec{p},2m)\;, \tag{8.79}$$

where g_1 and g_3 are the two Lorentz-invariant coupling constants. This amplitude incorporates parity conservation in the decay in the sense that when the helicity vector is replaced by the canonical vector (i.e. replace λ by m_1), the amplitude is invariant under parity operation [this can be shown by using Eqs. (8.30) and (8.33)]. Note that the amplitude A as it appears in Eq. (8.79) is not parity-invariant, simply because the helicity λ changes sign under parity operation.

Let us express the amplitude A in the JRF:

$$A\sim g_1\,e^{i\,*}(\vec{p}_1,\lambda)\;p_1^j\;e_{ij}(2m)+g_3\,we^0(\vec{p}_1,\lambda)\,p_1^i\,p_1^j\;e_{ij}(2m)\;. \tag{8.79a}$$

Using the same technique we have used in previous examples, we expand e_{ij} in terms of the e'_{ij} whose quantization axis points along the momentum $\vec{p}_1$. Afterwards, we use the relation (8.71) and [use Eqs. (8.21)]

$$\vec{e}^{\,*}(\vec{p}_1, \lambda) \cdot \vec{e}^{\,\prime}(m_1) = \left[\lambda^2 + \frac{E_1}{w_1}(1 - \lambda^2)\right] \delta_{\lambda, m_1} \tag{8.80}$$

to cast the amplitude A into the following form:

$$A \sim B_\lambda \, D^{(2)\,*}_{m\lambda}(\phi, \theta, 0) \,, \tag{8.81}$$

where

$$\begin{aligned} B_\pm &= \frac{1}{\sqrt{2}} g_1 \, p_1 \\ B_0 &= \sqrt{\frac{2}{3}} \left[g_1 \, p_1 \frac{E_1}{w_1} + g_3 \, p_1^3 \frac{w}{w_1} \right] \,. \end{aligned} \tag{8.82}$$

Comparison of Eq. (8.81) with Eq. (8.65) shows that B_λ may be set equal to F^J_λ, which has the following expansion in terms of the partial-wave amplitudes $a^J_\ell \, (\ell = 1 \text{ or } 3)$:

$$\begin{aligned} F_\pm &= \sqrt{\frac{3}{10}} \, a_1 + \sqrt{\frac{1}{5}} \, a_3 \\ F_0 &= \sqrt{\frac{2}{5}} \, a_1 - \sqrt{\frac{3}{5}} \, a_3 \,. \end{aligned} \tag{8.83}$$

where the superscripts J have been suppressed for brevity. Equations (8.82) and (8.83) can be used to solve for a_1 and a_3 in terms of g_1 and g_3 by setting $B_\lambda = F_\lambda$

$$\begin{aligned} a_1 &= \sqrt{\frac{1}{15}} \left[g_1 \, p_1 \left(\frac{2E_1 + 3w_1}{w_1} \right) + 2 g_3 \, p_1^3 \, \frac{w}{w_1} \right] \\ a_3 &= -\sqrt{\frac{2}{5}} \left[g_1 \, p_1 \left(\frac{E_1 - w_1}{w_1} \right) + g_3 \, p_1^3 \, \frac{w}{w_1} \right] \,. \end{aligned} \tag{8.84}$$

We have now exhibited clearly how the Lorentz invariant coupling constants are related to the partial-wave amplitudes. We see that in general it is not possible to define a Lorentz invariant coupling constant which corresponds to a single partial-wave amplitude. The reason is that the orbital angular momentum is a well-defined concept only in the parent-resonance rest frame (the JRF).

One may ask the following question: Is it possible in the tensor formalism to construct amplitudes corresponding to states of a pure orbital angular momentum? The answer is that one has to use the non-relativistic tensor formalism. Let us examine the following non-relativisitc amplitude:

$$A \sim b_1 \, e^{i\,*}(\lambda) \, p_1^j \, e_{ij}(2m) + b_3 \, e^{i\,*}(\lambda) \, T^{(3)}_{ijk}(p_1) \, e^{jk}(2m) \,, \tag{8.85}$$

where $T^{(3)}_{ijk}$ is the pure spin-3 tensor given in Eq. (8.53).

Let us proceed to the task of reformulating the above amplitude to the one similar to Eq. (8.65). Using the by-now standard technique, we re-express e_{ij} in terms of e'_{ij} as given in Eq. (8.69). For the first term in formula (8.85), we evaluate [use Eq. (8.17a)]

$$e^{i\,*}(\lambda)\, p_1^j\, e'_{ij}(2m') = p_1\,(101\lambda|2\lambda)\,\delta_{\lambda m'}$$

to obtain the result

$$b_1\, p_1\,(101\lambda|2\lambda)\; D^{(2)\,*}_{m\lambda}(\phi,\theta,0) \tag{8.86}$$

This shows that the first term in indeed proportional to the P-wave amplitude [see Eq. (8.67)]:

$$a_1 = \sqrt{\frac{5}{3}}\, b_1\, p_1 \ . \tag{8.87}$$

The second term can be reduced to the desired form by using the explicit expression for the third-rank tensor given in Eq. (8.53). However, it is instructive to use the following method, which is easily applicable to higher spin cases. Let us express the tensor in terms of the projection operator:

$$\begin{aligned} T^{(3)}_{ijk}(p_1) &= P^{(3)}_{ijk\ell mn}\, p_1^\ell\, p_1^m\, p_1^n \\ &= \sum_\mu e_{ijk}(3\mu)\; e^*_{\ell mn}(3\mu)\; p_1^\ell\, p_1^m\, p_1^n \ . \end{aligned} \tag{8.88}$$

If we assume that the z-component μ is defined along the direction of $\vec{p}_1$, then the tensor reduces to the following simple form [it should be emphasized that this form does not correspond to a general expression; see Eq. (8.53)]:

$$\begin{aligned} T^{(3)}_{ijk}(p_1) &= p_1^3\,(2010|30)(1010|20)\; e_{ijk}(3,0) \\ &= \sqrt{\frac{2}{5}}\, p_1^3\, e_{ijk}(3,0) \ , \end{aligned} \tag{8.88a}$$

where

$$\begin{aligned} e_{ijk}(3,0) &= \sum_\mu (1\mu 2\, {-\mu}|30)\; e_1(\mu)\; e_{jk}(2,-\mu) \\ &= -\sqrt{\frac{7}{5}} \sum_\mu (301\mu|2\mu)\; e_2(\mu)\; e^*_{jk}(2\mu) \ . \end{aligned} \tag{8.88b}$$

From the normalization (8.38), we see that

$$e^*_{jk}(2\mu)\; e'^{jk}(2m') = \delta_{\nu m'} \ . \tag{8.88c}$$

Note that this result follows because both the spin-2 wave functions have the same quantization axis. Substituting Eq. (8.88) into Eq. (8.85), we finally obtain for the second term

$$-\frac{\sqrt{14}}{5}\, b_3\, p_1^3\,(301\lambda|2\lambda)\; D^{(2)\,*}_{m\lambda}(\phi,\theta,0) \ , \tag{8.89}$$

which shows clearly that this term corresponds to a pure F-wave amplitude. Comparing this with Eq. (8.67), we may identify

$$a_3 = -\sqrt{\frac{2}{5}}\, b_3\, p_1^3 \; . \tag{8.90}$$

We have now completed the proof that the non-relativistic amplitude given in Eq. (8.85) corresponds to a sum of pure P- and F-wave amplitudes. It is clear how to write down amplitudes similar to (8.85) for different spin parity combinations for the parent resonance J. Let us make the following remarks concerning this type of amplitude. First, it gives the correct angular distribution in the JRF corresponding to a given orbital angular momentum. Second, it is "relativistically correct" in the sense that b_1 and b_3 can be expressed in terms of the Lorentz invariant coupling constants g_1 and g_3 [see Eqs. (8.82)]. In a phenomenological approach, it is clearly irrelevant as to which set of constants are used to describe the amplitude.

Suppose that the vector particle s decays into two pseudo-scalar mesons. In order to describe the over-all amplitude for the J decay into three pseudo-scalar mesons, one has to multiply Eq. (8.85) by a factor $\vec{k} \cdot \vec{e}\,(\lambda)$ and sum over the helicity λ, where $\vec{k}$ is the relative decay momentum in the sRF. Owing to the relation (8.17b), the net effect is to replace $e^{i*}(\lambda)$ by k^i in (8.85):

$$A \sim b_1 k^i p_1^j e_{ij}(2m) + b_3 k^i T^{(3)}_{ijk}(p_1) e^{jk}(2m) \; . \tag{8.91}$$

The Dalitz plot distribution is obtained by taking the absolute value of A and summing over m [see expression (8.57)]:

$$\sum_m |A|^2 \sim \phi^*_{ij}\, P^{(2)}_{ijk\ell}\, \phi_{k\ell} \; , \tag{8.92}$$

where

$$\phi^*_{ij} \sim b_1^*\, k_i\, p_{ij} + b_3^*\, k^\ell\, T^{(3)}_{\ell ij}(p_1) \tag{8.93}$$

and

$$P^{(2)}_{ijk\ell}\, \phi_{k\ell} \sim b_1\, T^{(2)}_{ij}(k, p_1) + b_3\, k^\ell\, T^{(3)}_{\ell ij}(p_1) \; . \tag{8.94}$$

Note that the second term in Eq. (8.94) is already symmetric and traceless in the indices i and j.

This exercise shows how to calculate the Dalitz-plot distribution function out of the non-relativistic amplitude we have constructed. It is important to realize that we are using momenta defined in two different rest frames; the $\vec{p}_1$ is given in the JRF, whereas the $\vec{k}$ is defined in the sRF. Of course, this is a consequence of the non-relativistic approach we have adopted here. For a more general discussion of the non relativistic tensor formalism, the reader is referred to Zemach [33].

One may consider Eq. (8.92) as defining the distribution in the internal angle, $\cos\theta_1 \sim \vec{k}\cdot\vec{p}_1$. Of course, the distribution in $\cos\theta_1$ may be derived within the helicity formalism. Note that the decay we consider here is a special case of the general sequential decays considered in the previous section. The desired distribution in $\cos\theta_1$ is obtained by integrating over all angles except $\cos\theta_1$ in Eq. (7.7):

$$I(\cos\theta_1) \sim \sum_\lambda g^{(2)}_{\lambda\lambda}\; [D^{(1)}_{\lambda 0}(\theta_1)]^2 \; , \tag{8.95}$$

where $g_{\lambda\lambda}^{(2)}$ is related to the partial-wave amplitudes $a_\ell^{(2)} (\ell = 1 \text{ or } 3)$ via

$$g_{\lambda\lambda}^{(2)} \sim \left| \sum_\ell (2\ell+1)^{\frac{1}{2}} \, a_\ell^{(2)} \, (\ell 0 1 \lambda | 2 \lambda) \right|^2 . \tag{8.96}$$

It can be checked that the angular distribution obtained using Eq. (8.92) is indeed identical to that given by the formula (8.95).

9 Tensor Formalism for Half-Integral Spin

In this section we wish to develop the tensor formalism for wave functions describing relativistic particles with half-integer spin. We construct explicitly the tensor wave functions and illustrate how they may be used to construct invariant amplitudes. We have seen in the previous section that it is advantageous to use the purely non-relativistic formalism, whereby one employs only the wave functions evaluated in the rest frames. For this reason, we shall first concentrate on the non-relativistic two-component tensor formalism and then generalize later to the four-component tensor formalism, which satisfies the Rarita-Schwinger conditions [31].

Our first task is to construct the spinor wave functions describing spin-1/2 particles as rest, and show how the Pauli matrices may be used to build up the rotationally invariant amplitudes. Next, we discuss the spin-3/2 wave function constructed out of a polarization vector and a spinor, and give the subsidiary conditions limiting the number of independent operators which are constructed from the spin wave functions. The pure tensors, normally obtained by the use of the projection operators, can also be constructed in the following manner

for half-integer spins. Let $T^{(n)}$ be a pure tensor for an integer spin n. The product of $T^{(n)}$ with a spinor will then correspond to a spin $n+1/2$ or $n-1/2$. As will be shown in this section, it is a simple matter to project out the spin $n+1/2$ or $n-1/2$ component from the product. This is the approach adopted by Zemach [33].

As for the relativistic wave functions for the half-integer spins, our starting point is the Dirac four-component formalism for spin-$\frac{1}{2}$ particles. Following Auvil and Brehm [32], we then combine the spin-$\frac{1}{2}$ wave function with the relativistic tensor wave function of the previous section to form explicitly the tensor wave funct8ions for arbitrary half-integer spins. We shall show that these wave functions satisfy the Rarita-Schwinger conditions. The explicit form of the relativistic projection operators corresponding to arbitrary half-integer spins has been given by Fronsdal [34].

9.1 Spin-1/2 states at rest

States corresponding to a particle of spin-1/2 at rest have two independent components, owing to the two values the z-component of the spin can take. In this case, the basis vectors may be given by the spinors or two dimensional column vectors:

$$\chi(+\tfrac{1}{2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi(-\tfrac{1}{2}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{9.1}$$

States $\chi(m)$ are, of course the analogue of the ket vectors $|jm\rangle$ with $j = 1/2$ [see Eqs. (1.2)].

It is well known that the representation of the angular momentum satisfying Eqs. (1.2) in the spinor basis is given by

$$\vec{J} = \frac{1}{2}\,\vec{\sigma} , \tag{9.2}$$

where σ_i is the Pauli matrix:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{9.3}$$

In fact, once the spin-1/2 basis vectors are given by Eqs. (9.1), the corresponding angular momentum matrices can be *derived* using Eqs. (1.2); the result can be expressed as in Eq. (9.2). this is to be contrasted with the approach we have taken in the previous section. There we have started out with 3×3 rotation matrices and, by considering the infinitesimal rotations, found the 3×3 matrix representation of the angular momentum, and then "derived" the corresponding basis vectors, which turned out to be the polarization vectors.

Under rotation, states $\chi(m)$ transform according to [see Eq. (1.5)]

$$U[R(\alpha,\beta,\gamma)]\,\chi(m) = \sum_{m'} D^{\frac{1}{2}}_{m'm}(\alpha,\beta,\gamma)\,\chi(m')\ . \tag{9.4}$$

The normalization and the completeness relation for $\chi(m)$ have the standard form:

$$\begin{aligned} \chi^\dagger(m)\chi(m') &= \delta_{mm'} \\ \sum_m \chi(m)\chi^\dagger(m) &= I\ , \end{aligned} \tag{9.5}$$

where $\chi^\dagger(m)$ is the Hermitian conjugate of $\chi(m)$, i.e. a two-dimensional *row* vector, and I is the 2×2 unit matrix.

We shall now derive an important property which is that the Pauli matrices $\vec{\sigma}$ may be considered a vector in the construction of invariant amplitudes. For the purpose, let us exhibit explicitly the 2×2 rotation matrix:

$$U[R(\alpha,\beta,\gamma)] = \exp\left(-i\frac{\alpha}{2}\sigma_3\right)\exp\left(-i\frac{\beta}{2}\sigma_2\right)\exp\left(-i\frac{\gamma}{2}\sigma_3\right)\ . \tag{9.6}$$

Each factor in this expression can be expanded by using the following general property:

$$\exp[-i\,\theta\,\vec{n}\cdot\vec{\sigma}] = \cos\theta - i\,\vec{n}\cdot\vec{\sigma}\ \sin\theta\ , \tag{9.7}$$

where $\vec{n}$ is a unit vector. We are now ready to prove the following relation:

$$U^{-1}[R]\sigma_i U[R] = R_{ij}\,\sigma_j\ . \tag{9.8}$$

If $U[R]$ is given by Eq. (9.6), R_{ij} is an element of the 3×3 matrix given in Eq. (8.13). It is a straightforward algebra to prove the relation (9.8) by using Eq. (9.7). The scalar product $\vec{\sigma}\cdot\vec{p}$ for an arbitrary momentum $\vec{p}$ is a rotational invariant in the sense that

$$\chi^\dagger(m_1)\,\vec{\sigma}\cdot\vec{p}\,\chi(m_2) = \chi'^\dagger(m_1)\,\vec{\sigma}\cdot\vec{p}\,'\ \chi'(m_2)\ , \tag{9.9}$$

where the prime indicate the rotated quantities:

$$\begin{aligned} \chi'(m) &= U[R]\,\chi(m) \\ p'_i &= R_{ij}\,p_j\ . \end{aligned} \tag{9.10}$$

The relation (9.9) is the analogue of Eq. (8.14) for the polarization vectors.

One important characteristic of the product $\vec{\sigma} \cdot \vec{p}$ is the fact that it is a pseudo-scalar, because the angular momentum vector $\vec{J} = 1/2\vec{\sigma}$ is a pseudo-vector. This allows us to write down, for example, a general amplitude for the decay of the $\Lambda(1115)$ via the weak interaction:

$$\mathcal{M} \sim \chi^{\dagger}_{N}(m')\,\mathcal{A}\,\chi_{\Lambda}(m)\;, \tag{9.11}$$

where the subscripts denote states corresponding to either a nucleon or a Λ, and $\mathcal{A}$ is given by

$$\mathcal{A} = a_0 + a_1\vec{\sigma} \cdot \vec{p}\,. \tag{9.12}$$

a_0 and a_1 are the coupling constants and $\vec{p}$ is the momentum of the nucleon in the Λ rest frame. Note that we have used in formula (9.11) the wave functions (or states) evaluated in two different rest frames. The situation here is identical to that discussed in the previous section for integer spins; in a phenomenological description, Eq. (9.11) is a perfectly valid expression in the sense that the constants a_0 and a_1 are merely linear combinations of the Lorentz invariant coupling constants. We shall give the explicit relations when we take up the discussion of relativistic wave functions.

9.2 Non-relativistic spin-3/2 and higher spin states

Wave functions corresponding to a particle of spin 3/2 may be constructed by coupling the polarization vector with a spinor:

$$\vec{\chi}(\tfrac{3}{2}m) = \sum_{m_1 m_2} (1m_1\tfrac{1}{2}m_2|\tfrac{3}{2}m)\vec{e}(m_1)\chi(m_2)\;. \tag{9.13}$$

We know that both $\vec{e}(m_1)$ and $\chi(m_2)$ transform under rotation in the standard way. The Clebsch-Gordan coefficient in Eq.. (9.13) ensures that the states $\vec{\chi}(3/2m)$ transform under rotation as those of a particle of spin-3/2 [use Eqs. (8.11), (9.4), and (A.14)]:

$$U[R(\alpha,\beta,\gamma)]\vec{\chi}(\tfrac{3}{2}m) = \sum_{m'} D^{\frac{3}{2}}_{m'm}(\alpha,\beta,\gamma)\vec{\chi}(\tfrac{3}{2}m')\;. \tag{9.14}$$

States $\vec{\chi}(3/2m)$ are by definition three-vectors, each component of which is a spinor. Therefore, there are six independent components for $\vec{\chi}(3/2m)$. On the other hand, states of a pure spin-3/2 particle have only four independent components, so that $\vec{\chi}(3/2m)$ of Eq. (9.13) ought to satisfy two subsidiary conditions. It is a matter of straightforward algebra to prove that the desired conditions assume the following form:

$$\vec{\sigma} \cdot \vec{\chi}(\tfrac{3}{2}m) = 0\;. \tag{9.15}$$

The wave functions $\vec{\chi}(3/2m)$ have the following simple normalization:

$$\vec{\chi}^{\dagger}(\tfrac{3}{2}m) \cdot \vec{\chi}(\tfrac{3}{2}m') = \delta_{mm'}\;, \tag{9.16}$$

where $\vec{\chi}^{\dagger}$ is a three-vector with each component being a two-dimensional row vector. The outer product summed over all spin states defines, as before, the spin-3/2 projection operator

$$P^{(\frac{3}{2})}_{ij} = \sum_{m} \chi_i(\tfrac{3}{2}m)\chi^{\dagger}_j(\tfrac{3}{2}m) \tag{9.17}$$

with the normalization given by

$$\sum_k P_{ik}^{(\frac{3}{2})} P_{kj}^{(\frac{3}{2})} = P_{ij}^{(\frac{3}{2})} \ . \tag{9.18}$$

By definition, the projection operator has the property

$$P_{ij}^{(\frac{3}{2})\dagger} = P_{ji}^{(\frac{3}{2})} \ . \tag{9.19}$$

Because of the condition (9.15), it satisfies, in addition,

$$\sigma_i \, P_{ij}^{(\frac{3}{2})} = 0 \ . \tag{9.20}$$

The following form clearly satisfies Eq. (9.20):

$$P_{ij}^{(\frac{3}{2})} \ = \delta_{ij} - \frac{1}{3}\sigma_i\sigma_j \ . \tag{9.21a}$$

It is simple to show that this form also satisfies Eqs. (9.18) and (9.19). Equation (9.21a) may be recast into the form:

$$P_{ij}^{(\frac{3}{2})} \ = \frac{2}{3}\left[\delta_{ij} + \frac{i}{2}\varepsilon_{ikj}\,\sigma_k\right] \ . \tag{9.21b}$$

We shall give later a general expression for half-integer spin projection operators.

Let us go on to a discussion of spin-5/2 wave functions. They may be construed by coupling spin-2 wave functions with the spinor:

$$\chi_{ij}(\tfrac{5}{2}m) = \sum_{m_1 m_2} (2m_1 \tfrac{1}{2}m_2|\tfrac{5}{2}m) e_{ij}(m_1)\chi(m_2) \ . \tag{9.22}$$

By construction, this wave function has the correct property under rotation:

$$U[R(\alpha,\beta,\gamma)]\chi_{ij}(\tfrac{5}{2}m) = \sum_{m'} D^{\frac{5}{2}}_{m'm}(\alpha,\beta,\gamma)\chi_{ij}(\tfrac{5}{2}m') \ . \tag{9.23}$$

Because our wave functions have been constructed by coupling to the maximum possible spin, it can be shown that the following expression is equivalent to Eq. (9.22):

$$\chi_{ij}(\tfrac{5}{2}m) \ = \sum_{m_1 m_2} (\tfrac{3}{2}m_1 1 m_2|\tfrac{5}{2}m)\, \chi_i(\tfrac{3}{2}m_1)\, e_j(m_2) \ . \tag{9.22a}$$

Now, we are ready to enumerate the supplementary conditions on χ_{ij} limiting the number of independent components to six. From Eq. (9.22) we see that χ_{ij} is symmetric and traceless in the indices i and j. On the other hand, we have an additional condition from (9.22a) and (9.15)

$$\sigma_i \, \chi_{ij}(\tfrac{5}{2}m) = 0 \ . \tag{9.24}$$

Let us multiply Eq. (9.24) by σ_j and sum over the index j. then, we obtain

$$(\delta_{ij} + i\varepsilon_{jik}\sigma_k)\chi_{ij}(\tfrac{5}{2}m) = 0 \ . \tag{9.25}$$

This relation shows us that, if χ_{ij} is symmetric, Eq. (9.24) automatically ensures that it is also traceless. It is thus clear that a symmetric χ_{ij} satisfying Eq. (9.24) has indeed six independent components.

The spin-5/2 wave functions are normalized according to

$$\sum_{ij} \chi^{\dagger}_{ij}(\tfrac{5}{2}m)\chi_{ij}(\tfrac{5}{2}m') = \delta_{mm'} \tag{9.26}$$

and give the spin-5/2 projection operator

$$P^{(\frac{5}{2})}_{ijk\ell} = \sum_{m} \chi_{ij}(\tfrac{5}{2}m)\chi^{\dagger}_{k\ell}(\tfrac{5}{2}m)\ . \tag{9.27}$$

This operator has the following properties:

$$\sum_{ab} P^{(\frac{5}{2})}_{ijab}\, P^{(\frac{5}{2})}_{abk\ell} \;=\; P^{(\frac{5}{2})}_{ijk\ell} \tag{9.28}$$

$$P^{(\frac{5}{2})\dagger}_{ijk\ell} \;=\; P^{(\frac{5}{2})}_{k\ell ij} \tag{9.29}$$

$$\sigma_i\, P^{(\frac{5}{2})}_{ijk\ell} \;=\; 0\ . \tag{9.30}$$

Of course, the projection operator is symmetric and traceless in the pairs of indices (i, j) and (k, ℓ). We give below explicit expressions for the spin-5/2 projection operator, as well as the general formula corresponding to arbitrary half-integral spins.

9.3 Non-relativistic spin projection operators

Consider the expression

$$P^{(\frac{5}{2})}_{ijk\ell} = \frac{3}{7}\sigma_m\, \sigma_n\, P^{(3)}_{mijnk\ell}\ , \tag{9.31}$$

where $P^{(3)}$ is the spin-3 projection operator [see Eq. (8.45b)]. It is clear that the projection operator (9.31) satisfies Eq. (9.29). It also satisfies Eq. (9.30), since

$$\begin{aligned} \sigma_i P^{(\frac{5}{2})}_{ijk\ell} &= \frac{3}{7}(2\delta_{mi} - \sigma_m\sigma_i)\, \sigma_n\, P^{(3)}_{mijnk\ell} \\ &= -\sigma_m\, P^{(\frac{5}{2})}_{mjk\ell}\ . \end{aligned}$$

It can also be shown that Eq. (9.31) obeys Eq. (9.28). The formula (9.31) is a special case of the general expression for projection operators given by Fronsdal [34]. The non-relativistic version on his formula reads as follows ($n \geq 1$):

$$P^{(n+\frac{1}{2})}_{i_1\cdots i_n j_1\cdots j_n} = \left(\frac{n+1}{2n+3}\right)\sigma_k\, \sigma_\ell\, P^{n+1}_{ki_1\cdots i_n \ell j_1\cdots j_n}\ , \tag{9.32}$$

where n is an integer and P^{n+1} is the projection operator for spin $(n + 1)$. Note that application of this formula for spin-3/2 gives immediately Eq. (9.21a).

After some algebra, Eq. (9.31) can be re-expressed in the following form

$$P^{(\frac{5}{2})}_{ijk\ell} = P^{(2)}_{ijk\ell} - \frac{1}{5}\left[\sigma_i\,\sigma_m\,P^{(2)}_{mjk\ell} + \sigma_j\,\sigma_m\,P^{(2)}_{imk\ell}\right] , \tag{9.33a}$$

where $P^{(2)}$ is the spin-2 projection operator [see Eq. (8.41)]. Or equivalently,

$$P^{(\frac{5}{2})}_{ijk\ell} = \frac{3}{5}P^{(2)}_{ijk\ell} + \frac{i}{5}\left[\varepsilon_{inm}\,\sigma_n\,P^{(2)}_{mjk\ell} + \varepsilon_{jnm}\,\sigma_n\,P^{(2)}_{imkl}\right] . \tag{9.33b}$$

Note that similarity of this formula with the form of the spin-3/1 projection operator given in Eq. (9.21*b*); these are the two simplest cases of the general formula give by Zemach [33]. Let us briefly outline his derivation. If we combine a pure tensor of integer spin n with a spinor, the product can at most describe spins $n+1/2$ or $n-1/2$. The operator projecting out the $(n+1/2)$ component is well known; combining it with the spin-n projection operator $P^{(n)}(n \geq 1)$,

$$P^{(n+\frac{1}{2})} = \frac{1}{2n+1}\left[n+1+\vec{\sigma}\cdot\vec{S}^{\,(n)}\right] P^{(n)} , \tag{9.34}$$

where $\vec{S}^{\,(n)}$ is the spin-n angular momentum operator given by

$$\vec{S}^{\,(n)} = \sum_{i=1}^{n} \vec{S}^{\,(i)} \tag{9.35}$$

and $\vec{S}^{\,(i)}$ is the spin-1 operator acting on the i^{th} vector index, its matrix element being given by Eq. (8.8). the reader can check that application of the formula (9.34) for spins 3/2 and 5/2 gives Eqs. (9.21*b*) and (9.33*b*), respectively.

Let us now turn to a discussion of the half-integral spin tensors. We shall give explicit expressions for the spin-3/2 and -5/2 tensors constructed out of an arbitrary three-vector $\vec{a}$:

$$\begin{aligned} \vec{T}^{(\frac{3}{2})}(a) &= \vec{a} - \frac{1}{3}\vec{\sigma}(\vec{\sigma}\cdot\vec{a}\,) \\ &= \frac{2}{3}\left[\vec{a} + \frac{i}{2}\vec{\sigma}\times\vec{a}\right] \end{aligned} \tag{9.36}$$

and

$$T^{(\frac{5}{2})}_{ij}(a) = qa_ia_j - \frac{a^2}{5}\delta_{ij} - \frac{1}{5}[a_i\sigma_j + a_j\sigma_i](\vec{\sigma}\cdot\vec{a}\,) . \tag{9.37}$$

Of course, these have been obtained by applying the projection operators (9.21*b*) and (9.33*b*). these tensors satisfy the constraint

$$\vec{\sigma}\cdot\vec{T}^{(\frac{3}{2})}(a) = 0 \tag{9.38a}$$

$$\sigma_i T^{(\frac{5}{2})}_{ij}(a) = 0 \tag{9.38b}$$

and $T^{(5/2)}_{ij}$ is, in addition, symmetric and traceless.

There exists an alternative way of constructing the spin tensors. It consists in multiplying the pure spin tensors by pseudoscalar $(\vec{\sigma}\cdot\vec{a}\,)$ *from the left*. The resulting tensors are then appropriate for describing particles of opposite parity. Let us write

$$Q^{(n+\frac{1}{2})}(a) = T^{(n+\frac{1}{2})}(a)(\vec{\sigma}\cdot\vec{a}\,) . \tag{9.39}$$

Note that the tensors Q satisfy the constraint (9.38b), because the T's obey the constraint. The explicit expressions for the spin-3/2 and 5/2 tensors are:

$$Q^{(\frac{3}{2})}(a) \ = \vec{a}\,(\vec{\sigma}\cdot\vec{a}\,) - \frac{a^2}{3}\vec{\sigma} \tag{9.36a}$$

and

$$Q^{(\frac{5}{2})}_{ij}(a) = a_i\,a_j\,(\vec{\sigma}\cdot\vec{a}\,) \ - \frac{a^2}{5}\delta_{ij}(\vec{\sigma}\cdot\vec{a}\,) - \frac{a^2}{5}[a_i\sigma_j + a_j\sigma_i]\;. \tag{9.37a}$$

The spin tensors Q may be obtained in the following alternative way:

$$Q^{(n+\frac{1}{2})}_{i_1\ldots i_n}(a) = \sigma_m T^{(n+1)}_{mi_1\cdots i_n}(a)\;a \tag{9.40}$$

where $T^{(n+1)}$ is the pure tensor corresponding to the integer spin $(n+1)$. Note that the tensors Q defined by Eq. (9.40) are transverse to $\vec{\sigma}$; this can be shown by following the same argument used in the discussion of Eq. (9.31). One can prove that the tensors (9.40) are identical to those defined by formula (9.39), using the general expression (9.32) for the half-integral spin projection operators [for this proof, one needs in addition the formula (3.28) of Zemach [33]]. the proof is simple for spin-3/2 and 5/2; one merely needs to contract the spin-2 and spin-3 tensors given in Eqs. (8.52) and (8.53) with $\vec{\sigma}$, to obtain the desired results (9.36a) and (9.37a).

Later, we shall illustrate with simple examples the uses of the spin tensors given in this section.

9.4 Dirac formalism for spin-1/2 states

We adopt the four-component Dirac formalism to describe the relativistic spin-1/2 states. In the rest frame, the two-component spinors $\chi(m)$ are generalized to the four-component spinors $u(0,m)$:

$$U(0,m) = \begin{pmatrix} \chi(m) \\ 0 \end{pmatrix}\;. \tag{9.41}$$

Let us suppose that a spin-1/2 particle with mass w has momentum $\vec{p}$ and energy E in an arbitrary frame. The boost operator which takes the rest-state wave function $u(0,m)$ to that of momentum $\vec{p}$ can be written

$$\mathcal{D}[l(\vec{p}\,)] = \begin{pmatrix} \cosh\alpha/2 & \vec{\sigma}\cdot\hat{p}\sinh\alpha/2 \\ \sigma\cdot\hat{p}\sinh\alpha/2 & \cosh\alpha/2 \end{pmatrix}\;. \tag{9.42}$$

where $\hat{p}$ is the unit vector along $\vec{p}$ and $\tanh\alpha = p/E$. Then, the inverse of the boost operator is given by

$$\mathcal{D}^{-1}[L(\vec{p}\,)] = \mathcal{D}L(-\vec{p}\,)]\;. \tag{9.43}$$

The formula (9.42) is the analogue of the boost operator (2.11) for arbitrary spin. However, it is different in one important respect; the operators (2.11) are unitary, whereas the "boosts" given by expression (9.42) are *not* unitary. In fact, there are two different ways of representing the homogeneous Lorentz group. One is the infinite-dimensional *unitary* representation, in which the infinitesimal generators are given by the Hermitian operators $\vec{J}$ and $\vec{K}$ [see Eq. (2.11)]. This is the representation we have used in Section 2. The other is the finite-dimensional non-unitary representation, where the generators of the boosts $\vec{K}$ may be given by either $+i\vec{J}$ or $-i\vec{J}$ so that $\vec{K}$ is not Hermitian if $\vec{J}$ is Hermitian. The operator defined in expression (9.42) corresponds to the second representation, in which the boost generators corresponding to both $+i\vec{\sigma}/2$ and $-i\vec{\sigma}/2$ have been employed. The reader is referred to Froissart and Omnès [8] for a discussion of these topics. It should be mentioned, however, that the form of the boost operator we have adopted here is not the same as that given in their article.

In analogy to the formula (2.12), the boosts (9.42) along an arbitrary direction can be re-expressed in terms of the boosts along the z-axis:

$$\mathcal{D}[L(\vec{p}\,)] = \mathcal{D}[\mathring{R}]\mathcal{D}[L_z(p)]\mathcal{D}^{-1}[\mathring{R}] \; , \tag{9.44}$$

where $\mathring{R}$ is the usual rotation which takes the z-axis into the direction of the momentum $\vec{p}$,

$$\mathcal{D}[\mathring{R}] = \begin{pmatrix} U[\mathring{R}] & 0 \\ 0 & U[\mathring{R}] \end{pmatrix} \tag{9.45}$$

and $U[\mathring{R}]$ is the 2×2 unitary matrix given in Eq. (9.6). It is easy to check the relation (9.44) by using Eq. (9.8). We are now ready to define the canonical and helicity states in the four-component formalism:

$$\begin{aligned} u(\vec{p},m) &= \mathcal{D}[L(\vec{p}\,)]u(0,m) \\ &= \mathcal{D}[\mathring{R}]\mathcal{D}[L_z(p)]\mathcal{D}^{-1}[\mathring{R}]u(0,m) \end{aligned} \tag{9.46}$$

and

$$\begin{aligned} u(\vec{p},\lambda) &= \mathcal{D}[L(\vec{p}\,)]\mathcal{D}[\mathring{R}]u(0,m) \\ &= \mathcal{D}[\mathring{R}]\mathcal{D}[L_z(p)]u(0,m) \; . \end{aligned} \tag{9.47}$$

Using the explicit expression (9.42) for the boost operator, the canonical states (9.46) can be cast into the familiar form

$$u(\vec{p},m) = \left[\frac{E+w}{2w}\right]^{\frac{1}{2}} \begin{pmatrix} \chi(m) \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+w}\chi(m) \end{pmatrix} \; . \tag{9.46a}$$

The rotational property (9.4) for the two-component spinors assures that the canonical and helicity states have the correct rotational property:

$$\mathcal{D}[R]u(\vec{p},m) = \sum_{m'} D^{\frac{1}{2}}_{m'm}(R)u(R\vec{p},m') \tag{9.48}$$

and

$$\mathcal{D}[R]u(\vec{p},\lambda) = u(R\vec{p},\lambda) \ . \tag{9.49}$$

The helicity states are related to the canonical states via the usual relation,

$$U(\vec{p},\lambda) = \sum_m D^{\frac{1}{2}}_{m\lambda}(\overset{\circ}{R})u(\vec{p},m) \ . \tag{9.50}$$

Now, the states of spin-1/2 particle have two degrees of freedom. This means that, if the four-component spinors are to describe spin-1/2 states, there has to be a supplementary condition relating the upper two components to the lower two components in the spinor. This supplementary condition is in fact the well-known Dirac equation:

$$(\gamma^\mu p_\mu - w)u(\vec{p},m) = 0 \ , \tag{9.51}$$

where the γ^μ's are the familiar 4×4 matrices satisfying the anti-commutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \ = 2g^{\mu\nu} \tag{9.52a}$$

and are given by

$$\gamma^0 = \gamma_0 \ = \begin{pmatrix} 1^0 & \\ 0 & -1 \end{pmatrix} \tag{9.52b}$$

$$\gamma^i = -\gamma_i \ = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{9.52c}$$

$$\gamma^5 = \gamma_5 \ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{9.52d}$$

$$\gamma^\mu = \gamma^+_\mu = \gamma^{-1}_\mu, \quad \gamma^{\mu\dagger} \ = \gamma^0\gamma^\mu\gamma^0, \quad \gamma^5 = -\gamma^0\gamma^5\gamma^0 \ . \tag{9.52e}$$

Using Eqs. (9.46a), (9.52b), and (9.52c), one can easily verify the Dirac equation (9.51).

Let us define the "adjoint" spinor,

$$\bar{u}(\vec{p},m) = u^\dagger(\vec{p},m)\gamma^0 \ , \tag{9.53}$$

which obeys the adjoint Dirac equation

$$\bar{u}(\vec{p},m)(\gamma^\mu p_\mu - w) = 0 \ . \tag{9.54}$$

Note that the boost operator (refeq9.42) may be expressed as

$$\mathcal{D}[L(\vec{p}\,)] = \mathcal{D}^\dagger[L(\vec{p}\,)] = \gamma^0\mathcal{D}^{-1}[L(\vec{p}\,)]\gamma^0 \ , \tag{9.55}$$

so that the adjoint spinor takes the form

$$\bar{u}(\vec{p},m) \ = \bar{u}(0,m)\mathcal{D}^{-1}[L(\vec{p}\,)] \ . \tag{9.53a}$$

In terms of the adjoint spinors, the normalization condition may be given by

$$\bar{u}(\vec{p},m)u(\vec{p},m') = \delta_{mm'} \tag{9.56}$$

and the projection operator can be defined

$$\begin{aligned}\Lambda_+(p) &= \sum_m u(\vec{p},m)\bar{u}(\vec{p},m) \\ &= \frac{1}{2w}(\gamma^\mu p_\mu + w) \ ,\end{aligned} \tag{9.57}$$

where one has used Eq. (9.53a) to obtain the explicit expression.

Let us now turn to a discussion of an important property of the γ^μ matrices:

$$\mathcal{D}^{-1}[\Lambda]\gamma^\mu\mathcal{D}[\Lambda] = \Lambda^\mu_\nu\gamma^\nu \ , \tag{9.58}$$

where Λ denotes an arbitrary Lorentz transformation. If Λ represents a rotation, one can readily prove this formula by using Eq. (9.8) for the Pauli matrices. One can also easily prove it for the case when Λ represents a pure Lorentz transformation. The reader may verify this for a Lorentz transformation along the z-axis by using the relations (2.6) and (9.42). The property (9.58) implies that the combination

$$\bar{u}(\vec{p}\,',m')\gamma^\mu u(\vec{p},m)$$

behaves like a four-vector under Lorentz transformations. This allows one to mix γ^μ with any four-momentum to construct Lorentz invariant amplitudes. One may think of this as the relativistic generalization of the property (9.9) for the Pauli matrices.

Let us give the transformation property of the four-component spinors under parity and time-reversal operations. Using Eqs. (3.9) and (3.10), we obtain

$$\Pi\, u(\vec{p},m) = \eta u(-\vec{p},m) = \eta\gamma^0 u(\vec{p},m) \tag{9.59}$$

$$\mathbb{T}\, u(\vec{p},m) = (-)^{\frac{1}{2}-m}u(-\vec{p},-m) = \gamma^3\gamma^1 u^*(\vec{p},m) \ , \tag{9.60}$$

where η is the intrinsic parity of the spin-1/2 particle. The last expressions in (9.59) and (9.60) can be checked by using Eqs. (9.46a), (9.52b), and (9.52c). The product $\gamma^3\gamma^1$ in Eq. (9.60) represents a rotation by π around the y-axis, which is associated with the action of $\mathbb{T}$ as explained in Section 3; note that

$$\gamma^3\gamma^1 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} = \begin{pmatrix} \exp\left(-i\frac{\pi}{2}\sigma_2\right) & 0 \\ 0 & \exp\left(-i\frac{\pi}{2}\sigma_2\right) \end{pmatrix} . \tag{9.61}$$

The adjoint spinors transform under Π and $\mathbb{T}$ according to

$$\bar{u}(\vec{p},m)\Pi^\dagger = \eta\bar{u}(\vec{p},m)\gamma^0 \tag{9.59a}$$

$$\bar{u}(\vec{p},m)\mathbb{T}^\dagger = \bar{u}^*(\vec{p},m)\gamma_1\gamma_3 \ . \tag{9.60a}$$

The γ^5 matrix in Eq. (9.52d) is a pseudoscalar in the sense that the combination

$$\bar{u}(\vec{p}\,',m')\gamma^5 u(\vec{p},m)$$

reverses sign under parity transformation [see Eq. (9.52e)].

Let us now come back to the discussion of the $\Lambda(1115)$ decay. We may write for the explicitly covariant decay amplitude,

$$m \sim \bar{u}_N(\vec{p}, m') B u_\Lambda(\vec{q}, m) \; , \tag{9.62}$$

where the first (second) spinor corresponds to that of the nucleon (the Λ), and B has the general expression

$$B = g_0 + g_1 \gamma^5 \; , \tag{9.63}$$

where g_0 and g_1 are the Lorentz invariant coupling constants. Evaluating the expression (9.62) in the Λ rest frame, we can find the relationship between the relativistic and non-relativistic coupling constants [compare expressions (9.11) and (9.62)]. They are

$$\begin{aligned} a_0 &= \left(\frac{E+w}{2w} \right)^{\frac{1}{2}} g_0 \\ a_1 &= -\frac{g_1}{[2w(E+w)]^{\frac{1}{2}}} \; , \end{aligned} \tag{9.64}$$

where w is the mass of the nucleon and E is its energy in the Λ rest frame.

9.5 Relativistic spin-3/2 and higher spin states

Wave functions corresponding to relativistic particles of spin $j(= n+1/2, n = \text{integer})$ can be constructed by coupling the spin-n tensor of the previous section with the four-component spinor. The rotational property (9.48) for the spinors and the similar property for the integer spin tensors [e.g. Eq. (8.34)] assure us that the desired wave functions can be written

$$u_{\mu_1 \cdots \mu_n}(\vec{p}, jm) = \sum_{m_1 m_2} (n m_1 \tfrac{1}{2} m_2 | jm) \, e_{\mu_1 \cdots \mu_n}(\vec{p}, n m_1) \, u(\vec{p}, m_2) \; , \tag{9.65}$$

where $e(\vec{p}, nm_1)$ is the spin-n tensor to be constructed in the manner described in the previous section. Of course, the adjoint wave function is constructed in the same way by coupling $e^*(\vec{p}, nm_1)$ with $\bar{u}(\vec{p}, m_2)$. Under rotations, the spin-j wave functions transform according to

$$\mathcal{D}[R] u(\vec{p}, jm) = \sum_m D^{-1}_{m'm}(R) u(R\vec{p}, jm') \; . \tag{9.66}$$

The wave functions given in Eq. (9.65) are, of course, the canonical wave functions. The helicity states may be constructed by merely replacing the m's in Eq. (9.65) by the λ's. It can be shown that the resulting helicity states are correctly related to the canonical states via

$$e(\vec{p}, j\lambda) = \sum_m D^j_{m\lambda}(\overset{\circ}{R}) e(\vec{p}, jm) \; , \tag{9.67}$$

where $\overset{\circ}{R}$ is as before the rotation which takes the z-axis into the direction of $\vec{p}$.

The spin-j wave function (9.65) is a four-component spinor with the four-vector indices $\mu_1 \cdots \mu_n$. Since it describes a state of spin j, it can have only $(2j+1)$ independent components. The desired supplementary conditions are just the Rarita-Schwinger equations [31]:

$$
\begin{aligned}
(\gamma^\mu p_\mu - w) u_{\mu_1 \cdots \mu_n} &= 0 && (9.68a) \\
u_{\cdots \mu_1 \cdots \mu_j \cdots} &= u_{\cdots \mu_j \cdots \mu_i \cdots} && (9.68b) \\
p^{\mu_1} u_{\mu_1 \cdots \mu_n} &= 0 && (9.68c) \\
\gamma^{\mu_1} u_{\mu_1 \cdots \mu_n} &= 0 && (9.68d) \\
g^{\mu_1 \mu_2} u_{\mu_1 \mu_2 \cdots \mu_n} &= 0 \ , && (9.68e)
\end{aligned}
$$

where w is the mass of the spin-j particle and p is its four-momentum. The relation (9.68d) is the relativistic generalization of the conditions (9.15) or (9.24). The reader may verify the relation (9.68d) for spin-3/2 by writing down the explicit expression for the spin-3/2 wave function and carrying out the necessary algebra. The condition (9.68e) is not an independent condition; it is in fact a consequence of the relations (9.68b) and (9.68d).

The spin-j wave functions u are normalized according to $(j = n + 1/2)$

$$
\bar{u}_{\mu_1 \cdots \mu_n}(\vec{p}, jm)\, u^{\mu_1 \cdots \mu_n}(\vec{p}, jm') = (-)^n \delta_{mm'} \tag{9.69}
$$

and they define the projection operator

$$
P^{(j)}_{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} = \sum_m u_{\mu_1 \cdots \mu_n}(\vec{p}, jm)\, \bar{u}_{\nu_1 \cdots \nu_n}(\vec{p}, jm) \ . \tag{9.70}
$$

This operator obviously satisfies all the conditions of Eqs. (9.68a)–(9.68e); in addition, it has the properties:

$$
P^{(j)}_{\mu_1 \cdots \mu_m\, \alpha_1 \cdots \alpha_n} P^{(j) \alpha_1 \cdots \alpha_n}_{\nu_1 \cdots \nu_n} = (-)^n P^{(j)}_{\mu_1 \cdots \mu_n\, \nu_1 \cdots \nu_n} \tag{9.71}
$$

$$
P^{(j)\dagger}_{\mu_1 \cdots \mu_n\, \nu_1 \cdots \nu_n} = \gamma^0 P^{(j)}_{\nu_1 \cdots \nu_n\, \mu_1 \cdots \mu_n} \gamma^0 \ . \tag{9.72}
$$

The explicit expression for the projection operator has been given by Fronsdal [34]. It may be written, in our notation,

$$
P^{(j)}_{\mu_1 \cdots \mu_n\, \nu_1 \cdots \nu_n} = -\left(\frac{n+1}{2n+3}\right) \Lambda_+(p) \gamma^\mu \gamma^\nu P^{(n+1)}_{\mu \mu_1 \cdots \mu_n\, \nu \nu_1 \cdots \nu_n} \ , \tag{9.73}
$$

where $\Lambda_+(p)$ is the spin-1/2 projection operator given in Eq. (9.57) and $P^{(n+1)}$ is the projection operator for the integral spin $(n+1)$. Evaluating this in the rest frame of the spin-j particle, one obtains the non-relativistic spin-3/2 projection operator:

$$
P^{(\frac{3}{2})}_{\mu\nu} = \Lambda_+(p) \left[\bar{g}_{\mu\nu} + \frac{1}{3}\, \bar{g}_{\mu\alpha}\, \gamma^\alpha\, \bar{g}_{\nu\beta}\, \gamma^\beta \right] \ . \tag{9.74}
$$

It can be shown easily that this operator satisfies the properties (9.71) and (9.72), as well as the conditions (9.68a)–(9.68e). Note that Eq. (9.74) reduces to Eq.
eqleq9.21a in the rest frame.

9.6 Applications

We give here a few simple examples to illustrate the uses of the tensor formalism we have developed so far. Through these examples, we wish to exhibit the connections between these and the helicity formalism of Section 3.

i) $\underline{\pi^+ p \to \Delta^{++}(1232) \to \pi^+ p}$

The invariant scattering amplitude for this process is proportional to

$$\mathcal{M}_{fi} \sim \sum_m \bar{u}(\vec{p}_f, m_f)\, p_f^\mu\, u_\mu(\vec{p}, \tfrac{3}{2}m)\, p_i^\nu\, u(\vec{p}_i, m_i) \;, \tag{9.75a}$$

where the subscripts i and f refer to the initial and final protons and $\vec{p}$ is the Δ^{++} momentum. By means of the spin-3/2 projection operator, the transition amplitude can be reduced to

$$\mathcal{M}_{fi} \sim \bar{u}(\vec{p}_f, m_f)\, p_f^\mu\, P_{\mu\nu}^{(\frac{3}{2})}\, p_i^\nu\, u(\vec{p}_i, m_i) \;. \tag{9.75b}$$

Using the expression (9.74), the square of the amplitude summed over the spin states of the protons can be given in an explicitly covariant form. However, we prefer to evaluate Eq. (9.75b) in the over-all c.m. system. Then, the amplitude takes the form, from Eq. (9.46a),

$$\mathcal{M}_{fi} \sim \left(\frac{E+w}{2w}\right) \chi^\dagger(m_f)\, \mathcal{A}\, \chi(m_i) \;, \tag{9.75c}$$

where w is the mass of the proton and E the proton energy, and

$$\begin{aligned} \mathcal{A} &\sim \vec{p}_f \cdot \vec{T}^{(\frac{3}{2})}(p_i) \\ &\sim \vec{p}_i \cdot \vec{p}_f + \frac{i}{2}\vec{\sigma} \cdot (\vec{p}_i \times \vec{p}_f) \;. \end{aligned} \tag{9.75d}$$

Here we have used the expression (9.36). Let us note that, aside from the energy dependent factor, the amplitude given in formula (9.75c) is precisely the one we would have obtained, had we started out with the non-relativistic formalism. The energy dependent factor is, of course, the consequence of the fact that the protons are *not* at rest in the over-all c.m.

Let us calculate the distribution in the scattering angle $\cos\theta \sim \vec{p}_i \cdot \vec{p}_f$. By taking the absolute square of the amplitude (9.75b) and performing the sum over the spin states,

$$\begin{aligned} \sum |\mathcal{M}_{fi}|^2 &\sim \operatorname{tr}\{\mathcal{A}\mathcal{A}^\dagger\} \\ &\sim 1 + 3\cos^2\theta \;, \end{aligned} \tag{9.76}$$

where we have suppressed the energy dependent factor.

The form of the amplitude (9.75a)–(9.75d) is such that it is invariant under parity transformations. The reader may verify this by writing down explicitly the expression for the spin-3/2 wave function [see Eq. (9.65)] and exhibiting its parity-transformation property [use the formulae (8.30) and (9.59)]. Suppose now that the $\Delta(1232)$ had spin-parity $3/2^-$. The modified scattering amplitude is, from formula (9.75b),

$$\mathcal{M}_{fi} \sim \bar{u}\, \gamma^5\, p_f^\mu\, P_{\mu\nu}^{(\frac{3}{2})}\, p_i^\nu\, \gamma^5\, u \tag{9.77a}$$

or, evaluating in the c.m. system,

$$\mathcal{M}_{fi} \sim \frac{1}{2w(E+w)}\chi^\dagger(m_f)B\chi(m_1) \; , \tag{9.77b}$$

where B is given by [see Eq. (9.36a)]

$$\begin{aligned} B &= (\vec\sigma\cdot\vec p_f)\mathcal{A}(\vec\sigma\cdot\vec p_i) \\ &\sim (\vec\sigma\cdot\vec p_f)\vec p_f\cdot\vec Q^{(\frac{3}{2})}(p_i) \\ &\sim (\vec p_f\cdot\vec p_i)(\vec\sigma\cdot\vec p_f)(\vec\sigma\cdot\vec p_i) - \frac{1}{3}p_f^2\,p_i^2 \; . \end{aligned} \tag{9.78}$$

Then, the distribution in the scattering angle is given by

$$\begin{aligned} \sum|\mathcal{M}_{fi}|^2 &\sim \operatorname{tr}\{BB^\dagger\} \\ &\sim 1+3\cos^2\theta \; , \end{aligned} \tag{9.79}$$

which is identical to the expression (9.76).

It is instructive to re-derive the angular distribution within the helicity formalism. Our starting point is the partial-wave expansion of the scattering amplitude given in Eq. (5.10). Limiting the expansion to a term corresponding to $J = 3/2$, we obtain for the angular distribution

$$I(\cos\theta) \sim \sum_{\lambda\lambda'}|F_\lambda F^*_{\lambda'}|^2[d^{\frac{3}{2}}_{\lambda\lambda'}(\theta)]^2 \tag{9.80}$$

where F_λ is the helicity decay amplitude of the $J = 3/2$ resonance. Owing to the parity conservation in the decay, we have

$$|F_+|^2 = |F_-|^2 \tag{9.81}$$

regardless of the intrinsic parity of the resonance. Then, the angular distribution given by

$$\begin{aligned} I(\cos\theta) &\sim [d^{\frac{3}{2}}_{++}(\theta)]^2 + [d^{\frac{3}{2}}_{+-}(\theta)]^2 \\ &\sim 1+3\cos^2\theta \; . \end{aligned} \tag{9.80a}$$

ii) <u>$3/2^+ \to 1/2^+ + 0^-$</u>

We shall consider this decay in terms of the helicity states for the nucleon. The decay amplitude may be written

$$\mathcal{M} \sim g\,\bar u(\vec p_1,\lambda)\,p_1^\mu\,u_\mu\,(\vec p, \tfrac{3}{2}m) \; , \tag{9.82}$$

where λ is the helicity of the nucleon and $\vec p_1$ is its momentum, and g is the invariant coupling constant. If the decay amplitude is evaluated in the $3/2^+$ rest frame, we obtain

$$\mathcal{M} \sim g\left[\frac{e_1+w_1}{2w_1}\right]^{\frac{1}{2}}\chi^\dagger(\lambda)\vec p_1\cdot\vec\chi(\tfrac{3}{2}m) \; , \tag{9.83}$$

where E_1 and w_1 are the energy and mass of the nucleon. Following exactly the same technique as that used for the integer-spin resonances, we re-express the spin-3/2 wave function in terms of that with the quantization axis along the momentum $\vec{p}_1$. Indicating this wave function by primes, we have

$$\vec{\chi}(\tfrac{3}{2}m) = \sum_{m'} D^{(\frac{3}{2})\,*}_{mm'}(\phi,\theta,0)\vec{\chi}'(\tfrac{3}{2}m') \;, \tag{9.84}$$

where (θ,ϕ) are the spherical angles describing the direction of $\vec{p}_1$. Substituting the expression (9.13) for $\vec{\chi}'$ in Eq. (9.84), we can cast Eq. (9.83) in the following form:

$$\mathcal{M} \sim F^{(\frac{3}{2})}_{\lambda}\, D^{(\frac{3}{2})\,*}_{m\lambda}(\phi,\theta,0) \;, \tag{9.85}$$

where

$$F^{(\frac{3}{2})}_{\lambda} \sim g \left[\frac{E_1+w_1}{2w_1}\right]^{\frac{1}{2}} p_1 (10\tfrac{1}{2}\lambda|\tfrac{3}{2}\lambda) \;. \tag{9.86}$$

The formula (9.85) is precisely that of the helicity formalism [see Eqs. (8.65) and (8.67)]. However, the tensor formalism gives further information; it gives the P-wave decay amplitude a_1 in terms of the invariant coupling constant g.

It is clear that the non-relativistic tensor formalism would have given us the identical result, as far as the angular dependence is concerned. The reason is that the parity conservation limits the decay amplitude to only *one* decay constant. Recall that we have encountered the same situation in connection with the decay $2^+ \to 1^- + 0^-$ discussed in the previous section. The situation becomes more complex, if the resonance decay allows more than one coupling constant, as we shall see in our next example.

iii) $\underline{3/2^- \to 3/2^+ + 0^-}$

The general amplitude for this decay process may be written, in the four-component formalism,

$$\mathcal{M} \sim g_0\, \bar{u}^{\mu}(\vec{p}_1,\tfrac{3}{2}\lambda)\, u_{\mu}(\vec{p},\tfrac{3}{2}m) + g_2\, \bar{u}^{\mu}(\vec{p}_1,\tfrac{3}{2}\lambda)\, p_{\mu}\, p_1^{\nu}\, u_{\nu}(\vec{p},\tfrac{3}{2}m) \;, \tag{9.87}$$

where the subscript 1 corresponds to the $3/2^+$ particle, λ is its helicity, and g_0 and g_2 are the invariant coupling constants. Evaluating $\mathcal{M}$ in the $3/2^-$ rest frame and using the rotated wave functions as before, the amplitude can be cast into the form

$$\mathcal{M} \sim B_{\lambda}\, D^{(\frac{3}{2})\,*}_{m\lambda}(\phi,\theta,0) \;, \tag{9.88}$$

where

$$\begin{aligned} B_{\pm\frac{3}{2}} &= \left[\frac{E_1+w_1}{2w_1}\right]^{\frac{1}{2}} g_0 \\ B_{\pm\frac{1}{2}} &= \frac{1}{3} g_0 \left[\frac{E_1+w_1}{2w_1}\right]^{\frac{1}{2}} \left(1+\frac{2E_1}{w_1}\right) + \frac{2}{3} g_2\, \frac{w}{w_1} p_1^2 \;, \end{aligned} \tag{9.89}$$

and w is the mass of the $3/2^-$ particle. Comparing this with the amplitude in the helicity formalism [see Eqs. (8.65) and (8.67)], one finds the connection between the invariant coupling constants and the partial-wave amplitudes a_0 and a_2 (corresponding to the S- and D-waves, respectively):

$$\begin{aligned} g_0 &= \left[\frac{2w_1}{E_1+2_1}\right]^{\frac{1}{2}}(a_0+a_2) \\ \frac{w}{w_1}p_1^2\, g_2 &= -\left(\frac{E_1}{w_1}-1\right)a_0-\left(\frac{E_1}{w_1}+2\right)a_2\ . \end{aligned} \tag{9.90}$$

If we wish to write down the amplitude in terms of the states corresponding to pure orbital angular momentum, then we have to use the non-relativistic tensor formalism. The situation here is similar to that of the decay $2^- \to 1^- + 0^-$ discussed in the previous section. The desired amplitude has the form

$$\mathcal{M} \sim b_0\vec{\chi}^\dagger(\tfrac{3}{2}\lambda)\cdot\vec{\chi}(\tfrac{3}{2}m) + b_2\chi_i^\dagger(\tfrac{3}{2}\lambda)T_{ij}^{(2)}(p_1)\chi_j(\tfrac{3}{2}m)\ . \tag{9.91}$$

By re-expressing this into the form (9.88), it can be shown that a_0 may be set equal to b_0 and a_2 to $-b_2p_1^2/3$.

Let us suppose now that the $3/2^+$ particle decays into a nucleon and an pion. Then, the amplitude (9.91) should be multiplied by a factor [see formula (9.83)]

$$\chi^\dagger(\lambda')\,\vec{k}\cdot\vec{\chi}(\tfrac{3}{2}\lambda)$$

and summed over the intermediate helicity λ (λ' is the nucleon helicity and $\vec{k}$ is the nucleon momentum in the $3/2^+$ rest frame). The over-all amplitude may be written

$$\mathcal{M}' \sim \chi^\dagger(\lambda')\,\vec{\phi}\cdot\vec{\chi}(\tfrac{3}{2}m)\ , \tag{9.92}$$

where

$$\phi_i \sim b_0\,T_i^{(\frac{3}{2})^\dagger}(k) + b_2\,T_j^{(\frac{3}{2})^\dagger}(k)T_{ji}^{(2)}(p_1)\ . \tag{9.93}$$

The square of the amplitude summed over the initial and final spin states takes the form

$$\sum|\mathcal{M}|^2 \sim tr\{\phi_i P_{ij}^{(\frac{3}{2})}\phi_j^\dagger\}\ , \tag{9.94}$$

where

$$P_{ij}^{(\frac{3}{2})}\phi_j^\dagger \sim b_0^*\,P_{ij}^{(\frac{3}{2})}T_j^{(\frac{3}{2})}(k) + b_2^*\,P_{ij}^{(\frac{3}{2})}T_{j\ell}^{(2)}(p_1)T_\ell^{(\frac{3}{2})}(k)\ . \tag{9.95}$$

This example illustrates how the projection operators and the spin tensors enter into the calculation of the square of amplitudes. Note that $\vec{p}_1$ is evaluated in the $3/2^-$ rest frame, while $\vec{k}$ is given in the $3/2^+$ rest frame.

The formula (9.94) defines the angular distribution in $\cos\theta_1 \sim \vec{k}\cdot\vec{p}_1$. Within the helicity formalism, the same angular distribution can be obtained by integrating over all angles except $\cos\theta_1$ in Eq. (7.7):

$$I(\cos\theta_1) \sim \sum_\lambda g_{\lambda\lambda}^{(\frac{3}{2})}\left\{[d^{\frac{3}{2}}_{\lambda+}(\theta_1)]^2 + [d^{\frac{3}{2}}_{\lambda-}(\theta_1)]^2\right\}\ , \tag{9.96}$$

where $g_{\lambda\lambda}^{(\frac{3}{2})}$ is given in terms of the partial-wave amplitudes $a_\ell^{(\frac{3}{2})}(\ell = 0 \text{ or } 2)$ by

$$g_{\lambda\lambda}^{(\frac{3}{2})} \sim \left| \sum_\ell (2\ell+1)^{\frac{1}{2}}\, a_\ell^{(\frac{3}{2})}\, (\ell 0 \tfrac{3}{2}\lambda | \tfrac{3}{2}\lambda) \right|^2 . \tag{9.97}$$

It can be shown that the distribution given in formula (9.94) is indeed identical to the expression (9.96).

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A D-Functions and Clebsch-Gordan Coefficients

We list here some useful formulae involving the rotation matrices $D^j_{m'm}(\alpha,\beta\gamma)$ and $d^j_{m'm}(\beta)$. In addition, we list a few relations involving the Clebsch-Gordan coefficients which have been used in the text. The explicit d functions for j up to three are given in Berman and Jacob [19].

For the rotation matrix, we use the definition as given in Rose [2], namely,

$$\begin{aligned} D^j_{m'm}(\alpha,\beta,\gamma) &= \langle jm'|e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}|jm\rangle \\ &= e^{i\,m'\alpha}\,d^j_{m'm}(\beta)\,e^{-i\,m\gamma}\ . \end{aligned} \tag{A.1}$$

By definition, the matrices $D^j_{m'm}$ are unitary and satisfy the group property:

$$\sum_k D^j_{m'k}(R)D^{j\,*}_{mk}(R) = \delta_{mm'} \tag{A.2}$$

$$D^j_{m'm}(R_2R_1) = \sum_k D^j_{m'k}(R_2)D^j_{km}(R_1)\ . \tag{A.3}$$

The D-functions are normalized according to

$$\int \mathrm{d}R\, D^{j_1\,*}_{\mu_1 m_1}(R)D^{j_2}_{\mu_2 m_2}(R) = \frac{8\pi^2}{2j_1+1}\delta_{j_1j_2}\delta_{\mu_1\mu_2}\delta_{m_1m_2}\ , \tag{A.4}$$

where $R = R(\alpha,\beta,\gamma)$ and $dR = d\alpha\, d\cos\beta\, d\gamma$.

The functions $d^j_{m'm}$ have the following symmetry properties:

$$d^j_{m'm}(\beta) = (-)^{m'-m}d^j_{mm'}(\beta) \tag{A.5}$$

$$d^j_{m'm}(\beta) = (-)^{m'-m}d^j_{-m'-m}(\beta) \tag{A.6}$$

$$d^j_{m'm}(\pi-\beta) = (-)^{j+m'}d^j_{m'-m}(\beta) \tag{A.7}$$

$$d^j_{m'm}(\pi) = (-)^{j-m}\delta_{m',-m}\ . \tag{A.8}$$

Owing to Eq. (A.6), the D-functions have the symmetry

$$D^{j\,*}_{m'm}(\alpha,\beta,\gamma) = (-)^{m'-m}D^j_{-m'-m}(\alpha\beta\gamma)\ . \tag{A.9}$$

One may use the identity

$$R(\pi+\alpha,\pi-\beta,\pi-\gamma) = R(\alpha,\beta,\gamma)R(0,\pi,0) \tag{A.10}$$

to show that

$$D^j_{m'm}(\pi+\alpha,\pi-\beta,\pi-\gamma) = (-)^{j-m}D^j_{m'-m}(\alpha,\beta,\gamma) \tag{A.11}$$

and

$$D^j_{m'm}(\pi+\phi,\pi-\theta,0) = e^{i\pi j}D^j_{m'-m}(\phi,\theta,0) \tag{A.12}$$

or taking $\phi\to-\phi$ and using Eq. (A.9)

$$D^j_{m'm}(\pi-\phi,\pi-\theta,0) = e^{i\pi j}\,(-)^{m'+m}D^j_{-m'm}(\phi,\theta,0) \tag{A.12a}$$

the spherical harmonics $Y^\ell_m(\theta,\phi)$ are related to the D-function via

$$D^{\ell\,*}_{m0}(\phi,\theta,0) = \sqrt{\frac{4\pi}{2\ell+1}}\,Y^\ell_m(\theta,\phi)\ . \tag{A.13}$$

The D-functions satisfy the following coupling rule:

$$D^{j_1}_{\mu_1 m_1} D^{j_2}_{\mu_2 m_2} = \sum_{j_3\mu_3 m_3} (j_1\mu_1\, j_2\mu_2|j_3\mu_3)(j_1 m_1\, j_2 m_2|j_3 m_3) D^{j_3}_{\mu_3 m_3}\ . \tag{A.14}$$

Or, equivalently,

$$D^{j_1}_{\mu_1 m_1} D^{j_3\,*}_{\mu_3 m_3} = \sum_{j_2\mu_2 m_2} \left(\frac{2j_2+1}{2j_3+1}\right) (j_1\mu_1 j_2\mu_2|j_3\mu_3)(j_1 m_1 j_2 m_2|j_3 m_3) D^{j_2\,*}_{\mu_2 m_2}\ . \tag{A.15}$$

Using Eqs. (A.4) and (A.15), one obtains

$$\int \mathrm{d}R\, D^{j_1}_{\mu_1 m_1}(R) D^{j_2}_{\mu_2 m_2}(R) D^{j_3\,*}_{\mu_3 m_3}(R) = \frac{8\pi^2}{2j_3+1}(j_1\mu_1 j_2\mu_2|j_3\mu_3)(j_1 m_1 j_2 m_2|j_3 m_3)\ . \tag{A.16}$$

The following relations involving Clebsch-Gordan coefficients have been used in Section 7. These formulae can be derived by using the recursion relations for Clebsch-Gordan coefficients [Edmonds [3], p. 39]. In terms of the shorthand notations

$$\tilde{L} = L(L+1) \quad \text{and} \quad \tilde{J} = J(J+1)$$

one may write

$$\frac{J-\frac{1}{2}L1|J\frac{1}{2})}{(J\frac{1}{2}L0|J\frac{1}{2})} = -\frac{2J+1}{\sqrt{\tilde{L}}} \qquad (\text{odd } L \geq 1) \tag{A.17}$$

$$\frac{(J\frac{3}{2}L0|J\frac{3}{2})}{(J\frac{1}{2}L0|J\frac{1}{2})} = 1-\frac{4\tilde{L}}{4\tilde{J}-3} \qquad (\text{even } L) \tag{A.18}$$

$$\frac{(J-\frac{3}{2}L2|J\frac{1}{2})}{(J-\frac{3}{2}L1|J-\frac{1}{2})} = \frac{J+\frac{1}{2}}{\sqrt{\tilde{L}-2}} \qquad (\text{even } L \geq 2) \tag{A.19}$$

$$\frac{(J1L0|J1)}{(J0L0|J0)} = 1-\frac{\tilde{L}}{2\tilde{J}} \qquad (\text{even } L) \tag{A.20}$$

$$\frac{(J-1L2|J1)}{(J0L0|J0)} = -\left[\frac{\tilde{L}}{\tilde{L}-2}\right]^{\frac{1}{2}} \qquad (\text{even } L \geq 2) \tag{A.21}$$

$$\frac{(J2L0|J2)}{(J0L0|J0)} = 1-\left(\frac{\tilde{L}}{2\tilde{J}}\right)\left(\frac{4\tilde{J}-\tilde{L}-2}{\tilde{J}-2}\right) \qquad (\text{even } L) \tag{A.22}$$

$$\frac{(J-2L4|J2)}{(J0L0|J0)} = \left[\frac{\tilde{L}(\tilde{L}-6)}{(\tilde{L}-2)(\tilde{L}-12)}\right]^{\frac{1}{2}} \qquad (\text{even } L \geq 4) \tag{A.23}$$

$$\frac{(J1L1|J2)}{(J0L0|J0)} = -\left[\frac{\tilde{L}-2}{\tilde{L}-6}\right]^{\frac{1}{2}}\left(3-\frac{\tilde{L}}{\tilde{J}}\right) \qquad (\text{even } L \geq 4)\ . \tag{A.24}$$

B Cross-Section and Phase Space

The purpose of this appendix is to show how one may define the cross-section and phase-space formulae, once the normalizations for the single-particle states have been fixed as in Eq. (2.20). Also listed here are a few explicit phase-space formulae, as they have been used in the main text. All the formulae listed here are, of course, extremely well known; we merely collect them here for ease of reference. The normalizations of one-particle states, as well as the conventions for cross-section and phase-space formulae, are the same as those given in Pilkuhn [11].

For simplicity of notation, we consider a reaction involving only spinless particles. Let us denote a reaction producing n particles in the final state, i.e.

$$a + b \to 1 + 2 + \ldots n \ . \tag{B.1}$$

In the over-all c.m. system, let w_0 be the c.m. energy, $\vec{p}_i$ the initial relative momentum, and $p_i(p_f)$ the over-all four-momentum in the initial (final) state. The differential cross-section corresponding to reaction (B.1) may be written, in terms of the invariant amplitude $\mathcal{M}_{fi}$,

$$\mathrm{d}\sigma = \frac{1}{4\mathcal{F}}|\mathcal{M}_{fi}|^2 \mathrm{d}\phi_n(1,2,\ldots,n), \qquad \mathcal{F} = \left[(p_a \cdot p_b)^2 - (w_a\, w_b)^2\right]^{1/2} \ , \tag{B.2}$$

where $\mathcal{F}$ is the flux factor, which in the over-all c.m. is given by

$$\mathcal{F} = p_i w_0 \tag{B.3}$$

and $\mathrm{d}\phi_n$ is the n-body phase space:

$$\mathrm{d}\phi_n(i \to 1,2,\ldots,n) = (2\pi)^4 \delta^{(4)}(p_1 + p_2 + \cdots + p_n - p_i) \prod_{k=1}^{n} \tilde{\mathrm{d}}p_k \ . \tag{B.4}$$

$\tilde{\mathrm{d}}p_k$ is the invariant volume element of the k^{th} particle as given in Eq. (2.22).

The phase-space formula may be broken up into two factors as follows:

$$\mathrm{d}\phi_n = \mathrm{d}\phi_\ell(i \to c, m+1, \cdots, n) \left(\frac{\mathrm{d}w_c^2}{2\pi}\right) \mathrm{d}\phi_m(c \to 1,2,\cdots,m) \ , \tag{B.5}$$

where $\ell + m = n + 1$ and c denotes a system consisting of particles 1 to m, its effective mass being w_c. After repeated application of Eq. (B.5) and using the explicit expression for the two-body phase space,

$$\mathrm{d}\phi_2 = \frac{1}{(4\pi)^2}\,\frac{p}{w}\,\mathrm{d}\Omega \ , \tag{B.6}$$

where w is the effective mass of the particles 1 and 2, and p and Ω denote the magnitude and direction of the relative momentum in the (1,2) rest frame, we may express the n-body phase-space succinctly as follows:

$$\mathrm{d}\phi_n = \frac{1}{2^n} \cdot \frac{(2\pi)^4}{(2\pi)^{3n}} \cdot \frac{p_0}{w_0}\,\mathrm{d}\Omega_0 \prod_{k=0}^{n-2} \{p\,\mathrm{d}w\,\mathrm{d}\Omega\}_k \ , \tag{B.7}$$

where the n-body phase space has been broken up into $n-1$ arbitrary two-body subsystems, each with effective mass w_k, relative momentum p_k, and direction Ω_k in the respective rest frame ($k = 0, 1, 2, \ldots, n-2$). We must require $n \geq 2$ and $\{\cdots\}_0 = 1$ to take care of the case with $k = 0$. Note that, for $n = 2$, Eq. (B.7) reduces to Eq. (B.6).

If $n = 3$, we obtain from Eq. (B.7),

$$\mathrm{d}\phi_3 = \frac{4}{(4\pi)^5} \frac{p_0}{w_0} \mathrm{d}\Omega_0 \{p \, \mathrm{d}w \, \mathrm{d}\Omega\} , \tag{B.8}$$

where we have dropped the subscript 1 from p_1, dw_1, and $d\Omega_1$. This formula is then the phase space appropriate for the reaction (5.25). Equation (B.8) may be changed into a different form by a simple change of variables:

$$\mathrm{d}\phi_3 = \frac{4}{(4\pi)^5} \mathrm{d}R \, \mathrm{d}E_1 \, \mathrm{d}E_2 , \tag{B.9}$$

where R stands for the Euler angles describing the orientation of the three-particle system, and $E_1(E_2)$ is the energy of the particle 1(2) in the over-all rest frame.

If $n = 4$, we see from Eq. (B.7) that

$$\mathrm{d}\phi_4 = \frac{16}{(4\pi)^8} \frac{p_0}{w_0} \mathrm{d}\Omega_0 \{p_s \, \mathrm{d}w \, \mathrm{d}\Omega\} \{p_1 \, \mathrm{d}w_s \, \mathrm{d}\Omega_1\} . \tag{B.10}$$

This is the formula corresponding to the process (7.1) in the notation described in Section 7. Formula (B.10) may be recast into a different form using Eqs. (B.8) and (B.9):

$$\mathrm{d}\phi_4 = \frac{16}{(4\pi)^8} \frac{p_0}{w_0} \mathrm{d}\Omega_0 \{w \, \mathrm{d}w \, \mathrm{d}R \, \mathrm{d}E_1 \, \mathrm{d}E_2\} , \tag{B.11}$$

which corresponds to the reaction (6.21).

We can now generalize the n-body phase formula Eq. (B.7) to include direct 3-body systems

$$\mathrm{d}\phi_n = \frac{1}{2^n} \cdot \frac{(2\pi)^4}{(2\pi)^{3n}} \cdot \frac{p_0}{w_0} \mathrm{d}\Omega_0 \prod_{k=0}^{n_2} \{p \, \mathrm{d}w \, \mathrm{d}\Omega\}_k \prod_{\ell=0}^{n_3} \{w' \, \mathrm{d}w' \, \mathrm{d}R \, \mathrm{d}E \, \mathrm{d}E'\}_\ell \tag{B.12}$$

where $n = n_2 + 2n_3 + 2 \geq 2$. $n_2 \geq 0$ is the number of 2-body subsytems, while $n_3 \geq 0$ is the number of 3-body subsytems in the problem. E_ℓ and E'_ℓ are the energies of any two particles of a 3-body system ℓ, evaluated in the 3-body rest frame. Again, we must require $n \geq 2$ and $\{\cdots\}_0 = 1$ to take care of the case with $k = 0$ or with $\ell = 0$.

C Normalization of two- and three-particle states

For simplicity of notation, we shall deal with spinless particles. Owing to Eqs. (2.20), two-particle states are normalized:

$$\langle \vec{p}_1'\vec{p}_2'|\vec{p}_1\vec{p}_2\rangle = \tilde{\delta}(\vec{p}_1' - \vec{p}_1)\,\tilde{\delta}(\vec{p}_2' - \vec{p}_2)\ , \tag{C.1}$$

where the invariant δ-function is given in Eq. (2.21). A system consisting of two momenta $\vec{p}_1$ and $\vec{p}_2$ may be described, in general, by one four-momentum p representing the sum of the four-momenta of particles 1 and 2 and Ω describing the orientation of the relative momentum in the (1,2) rest frame, i.e.

$$|p, \Omega\rangle = a\,|\vec{p}_1\vec{p}_2\rangle\ . \tag{C.2}$$

We adopt the normalization for this state as follows:

$$\langle p', \Omega'|p, \Omega\rangle = (2\pi)^4\delta^{(4)}(p' - p)\,\delta^{(2)}(\Omega' - \Omega)\ . \tag{C.3}$$

Let us multiply (C.1) and (C.3) by the invariant volume element $\tilde{d}p_1\,\tilde{d}p_2$ [see Eq. (2.22)], and integrate over these variables. Note that Eq. (C.1) gives 1, whereas Eq. (C.3) involves an integration of the following form [see Eq. (B.4)]

$$\int \delta^{(2)}(\Omega' - \Omega)\,\mathrm{d}\phi_2(1,2) = \frac{1}{(4\pi)^2}\frac{p}{w}$$

after using the formula (B.6). From Eq. (C.2), we see immediately that

$$a = \frac{1}{4\pi}\sqrt{\frac{p}{w}}\ , \tag{C.4}$$

Next, we turn to a discussion of the three-particle states, normalized according to

$$\langle p_1'\,p_2'\,p_3'|p_1\,p_2\,p_3\rangle = \prod_{i=1}^{3}\tilde{\delta}(p_i' - p_i)\ . \tag{C.5}$$

A system of three particles with momentum $\vec{p}_1, \vec{p}_2$, and $\vec{p}_3$ may be specified by a four-momentum p representing the sum of the three individual four-momenta, the Euler angles $R(\alpha,\beta,\gamma)$ describing the orientation in the rest frame, and E_1 and E_2, the energies of the particles 1 and 2, evaluated in the rest frame. Let us write

$$|P, R, E_1, E_2\rangle = b\,|\vec{p}_1\,\vec{p}_2\,\vec{p}_3\rangle \tag{C.6}$$

with the normalization

$$\langle P', R', E_1', E_2'|P, R, E_1, E_2\rangle = (2\pi)^4\delta^{(4)}(p' - p)\,\delta^{(3)}(R' - R)\,\delta(E_1' - E_1)\,\delta(E_2' - E_2)\ . \tag{C.7}$$

As with the two-particle system, we multiply Eqs. (C.5) and (C.7) by $d\tilde{p}_1\,d\tilde{p}_2\,d\tilde{p}_3$ and integrate over these variables. Equation (C.5) gives 1, while for Eq. (C.7) one needs to evaluate

$$\int \delta^{(3)}(R' - R)\,\delta(E_1' - E_1)\,\delta(E_2' - E_2)\,\mathrm{d}\phi_3(1,2,3) = \frac{4}{(4\pi)^5}\ ,$$

where one has used Eq. (B.9). From Eq. (C.6), we see that the normalization constant b is

$$b^{-1} = 8\pi^2\sqrt{4\pi}\ . \tag{C.8}$$