General Properties Of Dalitz Plots

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Abstract

This document derives some general properties of Dalitz plots. In particular, a derivation of the relation between the density along a decay band and the cosine of the decay angles is presented. It is also shown that given two identical particles, how the helicity angles of the two decay chains are related.

1 Properties of Dalitz Plots

We will consider the Dalitz plot as evaluated in the center of mass frame of the three daughter particles. Under that assumption, we have the following constraints:

$$\begin{aligned} M &= E_1 + E_2 + E_3 \\ 0 &= \vec{p}_1 + \vec{p}_2 + \vec{p}_3 \\ E_1 &= \sqrt{p_1^2 + m_1^2} \\ E_2 &= \sqrt{p_2^2 + m_2^2} \\ E_3 &= \sqrt{p_3^2 + m_3^2}. \end{aligned}$$

The Dalitz plot is made by plotting either the kinetic energy of one particle versus another, *e.g.* T_3 versus T_2, or by plotting the invariant mass of one pair of particles versus another pair, *e.g.* m_{13}^2 versus m_{12}^2.

The invariant mass of a pair of particles is written as:

$$\begin{aligned} m_{ij}^2 &= (E_i + E_j)^2 - (\vec{p}_i + \vec{p}_j)^2 \\ &= m_i^2 + m_j^2 + 2(E_iE_j - \vec{p}_i \cdot \vec{p}_j) \\ &= m_i^2 + m_j^2 + 2(E_iE_j - p_ip_j \cos\theta_{ij}). \end{aligned}$$

There is also a global constraint relating the seven masses in the problem:

$$M^2 = m_{12}^2 + m_{13}^2 + m_{23}^2 - m_1^2 - m_2^2 - m_3^2.$$

Consider now the addition of m_{12}^2 and m_{13}^2, where we replace $\vec{p}_3 = -\vec{p}_1 - \vec{p}_2$ and $E_3 = M - E_1 - E_2$. It is easily shown that

$$E_1 = \frac{M^2 + m_1^2 - m_{23}^2}{2M}$$

$$
\begin{aligned}
E_2 &= \frac{M^2 + m_2^2 - m_{13}^2}{2M} \\
E_3 &= \frac{M^2 + m_3^2 - m_{12}^2}{2M}.
\end{aligned}
$$

This confirms that the original statement on energy and invariant-mass squared being equivalent.

Now let us consider a situation where our particle M decays into m_{23} recoiling against particle 1. We want to define a coordinate system whose z-axis is along $\vec{p}_1$ and whose y-axis is normal to the decay plane of the three particles. We then want to boost along $\hat{z}$ into the rest frame of m_{23} and look at the angle between particle 2 and the $\hat{z}$ axis in that frame, $\cos\theta_2$. In the rest frame of M, the momentum of particle 2 is given as

$$
\vec{p}_2 = \begin{pmatrix} p_2 \sin\theta_{12} \\ 0 \\ -p_2 \cos\theta_{12} \end{pmatrix}
$$

where θ_{12} is the smallest angle between $\vec{p}_1$ and $\vec{p}_2$. The Lorentz boost into the rest frame of m_{23} (called the helicity frame) is given by :

$$
\begin{aligned}
\beta &= -\frac{p_1}{M - E_1} \\
\gamma &= \frac{M - E_1}{m_{23}} \\
\beta\gamma &= -\frac{p_1}{m_{23}}
\end{aligned}
$$

This then yields that

$$
\begin{aligned}
E_2^h &= \gamma E_2 - \beta\gamma p_2 \cos\theta_{12} \\
\vec{p}_2^{\,h} &= \begin{pmatrix} p_2 \sin\theta_{12} \\ 0 \\ -\gamma p_2 \cos\theta_{12} + \beta\gamma E_2 \end{pmatrix}
\end{aligned}
$$

For a given m_{23}, the magnitude of $\vec{p}_2^{\,h}$ and E_2^h are given as:

$$
\begin{aligned}
p_2^h &= \sqrt{\frac{(m_{23}^2 + m_3^2 - m_2^2)^2}{4m_{23}^2} - m_3^2} \\
E_2^h &= \sqrt{\frac{(m_{23}^2 - m_3^2 + m_2^2)^2}{4m_{23}^2}}
\end{aligned}
$$

Now, we can write down what the $\cos\theta_2$ is in the helicity frame. This can be simplified by inserting β, γ from above and noting that $p_2 \cos\theta_{12}$ can be replaced by: $(m_1^2 + m_2^2 + 2E_1E_2 - m_{12}^2)/p_1$

$$
\begin{aligned}
\cos\theta_2 &= \frac{-\gamma p_2 \cos\theta_{12} + \beta\gamma E_2}{p_2^h} \\
&= \frac{1}{p_2^h}\left[\frac{M - E_1}{m_{23}} \frac{m_{12}^2 - m_1^2 - m_2^2 - 2E_1E_2}{2p_1} - \frac{p_1}{m_{23}} E_2\right] \\
&= \frac{1}{2p_2^h p_1 m_{23}} \left[(M - E_1)(m_{12}^2 - m_1^2 - m_2^2 - 2E_1E_2) - 2p_1^2 E_2\right] \\
&= \frac{\left[-2m_{23}^2\right] m_{13}^2 + \left[m_{23}^2(m_2^2 + m_3^2 - m_{23}^2) + m_1^2(m_2^2 - m_3^2 + m_{23}^2) - M^2(m_2^2 - m_3^2 - m_{23}^2)\right]}{4Mp_2^h p_1 m_{23}}
\end{aligned}
$$

Which for a fixed value of m_{23}^2, p_1, and p_2^h implies that $\cos\theta_2$ is a linear function of m_{13}^2. We can now substitute in for the values of p_1 and p_2^h which leads to the following.

$$
\cos\theta_2 = \frac{\left[-2m_{23}^2\right] m_{13}^2 + \left[m_{23}^2(m_2^2 + m_3^2 - m_{23}^2) + m_1^2(m_2^2 - m_3^2 + m_{23}^2) - M^2(m_2^2 - m_3^2 - m_{23}^2)\right]}{\sqrt{\left[M^4 + (m_1^2 - m_{23}^2)^2 - 2M^2(m_1^2 + m_{23}^2)\right]\left[m_2^4 + (m_{23}^2 - m_3^2)^2 - 2m_2^2(m_{23}^2 + m_3^2)\right]}}
$$

2 Symmetries

Let us now assume that $m_2 = m_3 = m$. This means that we should expect symmetries around the line $m_{12}^2 = m_{13}^2$ in the plot. Let us explore this to see what we find. We will define two possible decay chains here:

$$\begin{aligned} X &\to A_{12}3 &\to 123 \\ X &\to B_{13}2 &\to 132 \end{aligned}$$

In the first decay chain, we have four angles: $(\theta_{gj}^A, \phi_{gj}^A, \theta_h^A, \phi_h^A)$, while in the second decay chain, we have four angles: $(\theta_{gj}^B, \phi_{gj}^B, \theta_h^B, \phi_h^B)$. Using the above equation for $\cos\theta$ and assuming first that $\vec{p}_3$ defines the $\hat{z}$ axis, we find that:

$$\begin{aligned} \cos\theta_h^A &= \frac{\left[-2m_{12}^2\right] m_{13}^2 + \left[m_{12}^2(m_2^2 + m_1^2 - m_{12}^2) + m_3^2(m_2^2 - m_1^2 + m_{12}^2) - M^2(m_2^2 - m_1^2 - m_{12}^2)\right]}{\sqrt{\left[M^4 + (m_3^2 - m_{12}^2)^2 - 2M^2(m_3^2 + m_{12}^2)\right]\left[m_2^4 + (m_{12}^2 - m_1^2)^2 - 2m_2^2(m_{12}^2 + m_1^2)\right]}} \\ \cos\theta_h^A &= \frac{\left[-2m_{12}^2\right] m_{13}^2 + \left[m_{12}^2(m^2 + m_1^2 - m_{12}^2) + m^2(m^2 - m_1^2 + m_{12}^2) - M^2(m^2 - m_1^2 - m_{12}^2)\right]}{\sqrt{\left[M^4 + (m^2 - m_{12}^2)^2 - 2M^2(m^2 + m_{12}^2)\right]\left[m^4 + (m_{12}^2 - m_1^2)^2 - 2m^2(m_{12}^2 + m_1^2)\right]}} \end{aligned}$$

Now, if we choose $\vec{p}_2$ to define the $\hat{z}$ axis, then we find that the angle of particle 1 in the helicity frame is:

$$\begin{aligned} \cos\theta_h^B &= \frac{\left[-2m_{13}^2\right] m_{12}^2 + \left[m_{13}^2(m_3^2 + m_1^2 - m_{13}^2) + m_2^2(m_3^2 - m_1^2 + m_{13}^2) - M^2(m_3^2 - m_1^2 - m_{13}^2)\right]}{\sqrt{\left[M^4 + (m_2^2 - m_{13}^2)^2 - 2M^2(m_2^2 + m_{13}^2)\right]\left[m_3^4 + (m_{13}^2 - m_1^2)^2 - 2m_3^2(m_{13}^2 + m_1^2)\right]}} \\ \cos\theta_h^B &= \frac{\left[-2m_{13}^2\right] m_{12}^2 + \left[m_{13}^2(m^2 + m_1^2 - m_{13}^2) + m^2(m^2 - m_1^2 + m_{13}^2) - M^2(m^2 - m_1^2 - m_{13}^2)\right]}{\sqrt{\left[M^4 + (m^2 - m_{13}^2)^2 - 2M^2(m^2 + m_{13}^2)\right]\left[m^4 + (m_{13}^2 - m_1^2)^2 - 2m^2(m_{13}^2 + m_1^2)\right]}} \end{aligned}$$

Finally, if we look along the symmetry axis where $m_{12}^2 = m_{13}^2$, then these two helicity angles are exactly equal.

Now let us assume that in the rest frame of X, we have some coordinate system whose unit vectors are given as $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$. If we now use these directions to define a $y - axis$ in the helicity frame as $\hat{y}$ is along $\hat{e}_z \times \hat{z}$. The question is now what is ϕ_h in the two frames? This angle is actually just the angle between the normal to the decay plane and the $\hat{y}$ axis as defined above. The normal to the decay plane is given as follows, where the angle is measured in the rest frame of X.

$$\begin{aligned} \hat{n} &= \frac{\vec{p}_1 \times \vec{p}_2}{p_1 p_2 \sin\theta_{12}} \\ &= \frac{\vec{p}_2 \times \vec{p}_3}{p_2 p_3 \sin\theta_{23}} \\ &= \frac{\vec{p}_3 \times \vec{p}_1}{p_3 p_1 \sin\theta_{31}} \end{aligned}$$

The $y - axis$ is given as:

$$\begin{aligned} \hat{y}^A &= \frac{-\vec{p}_3 \times \hat{e}_z}{p_3 \sin\theta_{3z}} \\ \hat{y}^B &= \frac{-\vec{p}_2 \times \hat{e}_z}{p_2 \sin\theta_{2z}} \end{aligned}$$

where again, the angle is measured in the rest-frame of X. The angle ϕ can now be determined as $\cos\phi = \hat{n} \cdot \hat{y}$.

$$\begin{aligned} \cos\phi_h^A &= \left(\frac{\vec{p}_2 \times \vec{p}_3}{p_2 p_3 \sin\theta_{23}}\right) \cdot \left(\frac{-\vec{p}_3 \times \hat{e}_z}{p_3 \sin\theta_{3z}}\right) \\ &= \frac{(\vec{p}_2 \cdot \vec{p}_3)(\vec{p}_3 \cdot \hat{e}_z) - (\vec{p}_2 \cdot \hat{e}_z)(\vec{p}_3 \cdot \vec{p}_3)}{p_2 p_3^2 \sin\theta_{23} \sin\theta_{3z}} \end{aligned}$$

Similarly, we can write that:

$$\cos\phi_h^B = \frac{(\vec{p}_2\cdot\vec{p}_2)(\vec{p}_3\cdot\hat{e}_z)-(\vec{p}_2\cdot\hat{e}_z)(\vec{p}_3\cdot\vec{p}_2)}{p_2^2 p_3\sin\theta_{23}\sin\theta_{2z}}.$$

It should be noted that both of these formulas have terrible pathologies. If any of the sin functions are zero, or if either p_2 or p_3 are zero. Other than this, there should be an exact one-to-one relation ship between ϕ_h^A and ϕ_h^B. Let's now restrict ourselves to the diagonal. Here we have that $m_2 = m_3 = m$, $p_2 = p_3 = p$, $E_2 = E_3 = E$ and $\cos\theta 12 = \cos\theta_{13}$. This means that $\theta_{12} = \theta_{13} = \theta$.

$$\cos\phi_h^A = \frac{\cos\theta_{23}\cos\theta_{3z}-\cos\theta_{2z}}{\sin\theta_{23}\sin\theta_{3z}} \quad (1)$$

$$\cos\phi_h^B = \frac{\cos\theta_{3z}-\cos\theta_{23}\cos\theta_{2z}}{\sin\theta_{23}\sin\theta_{2z}} \quad (2)$$

While the above are certainly true, there is an ambiguity in that both ϕ and $-\phi$ map to the same value of $\cos\phi$. To resolve this, we need to look more carefully and $\hat{n}$ and $\hat{y}$. Both of these are normal to $\hat{z}$, so our ϕ is actually the rotation angle around the $\hat{z}$ axis that moves $\hat{y}$ to $\hat{n}$. We also need to be careful about our definition of the normal in the two cases. In the case A_{12}, we define the normal to be

$$\hat{n}_A = \frac{\vec{p}_1\times\vec{p}_2}{|\,\vec{p}_1\times\vec{p}_2\,|}$$

$$\hat{n}_B = \frac{\vec{p}_1\times\vec{p}_3}{|\,\vec{p}_1\times\vec{p}_3\,|}$$

$$\phi_A^h = -\tan^{-1}\left(\frac{\hat{n}_A\cdot\hat{y}_A^h}{\hat{n}_A\cdot\hat{x}_A^h}\right)$$

$$\phi_A^h = -\tan^{-1}\left(\frac{\hat{n}_B\cdot\hat{y}_B^h}{\hat{n}_B\cdot\hat{x}_B^h}\right)$$

Finally, we should note that a given point in the Dalitz plot specifies the magnitudes of the three 3-momenta in the decay plane. It does **not** specify the orientation of the decay plane. We can apply any arbitrary rotation to the three particles we want, and we will still get the same entry in the Dalitz plot. This means that a particular point in the Dalitz plot does not translate into a unique θ_{gj} or ϕ_{gz} as it does for the angles in the helicity frame.