# PWA Formalism

M. Williams

Department of Physics
Carnegie Mellon University

Febuary 10, 2006

# Outline

# Outline

# Outline

# Outline

# Outline

# Outline

# Motivation

- Helicity formalism has awkward handling of final state particles with spin
- Helicity formalism is *NOT* relativistic
- We need to combine $s-, t-, u-$channel diagrams
- This formalism allows PWA parameters $\rightarrow$ coupling constants
- C++ makes the code easy to deal with

# General Technique

- Embed spin-j system, $(2j+1)$ independent elements, into higher dimension space of rank-j tensors, $4^j$ independent elements.
- Supplementary conditions, Rarita-Schwinger, reduce independent elements to the proper number.
- Why do this?
  - Transformation properties of spin-j system become that of rank-j tensors.
  - Forming Lorentz scalars becomes straight forward.

# Spin-1 States at Rest

Let $\vec{\epsilon}(\pm 1) = \mp \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$ and $\vec{\epsilon}(0) = \hat{z}$.

Using the rotation operator about the $k-$axis, $e^{-i\theta J_k}$, and considering infinitesimal rotations we get $J^k_{ij} = -i\epsilon_{kij}$.

It then follows that,

- $J_z\vec{\epsilon}(m) = m\vec{\epsilon}$
- $J^2\vec{\epsilon}(m) = 2\vec{\epsilon}(m)$
- $\vec{\epsilon}^*(m) \cdot \vec{\epsilon}(m') = \delta_{mm'}$

Thus, $\vec{\epsilon}(m)$ are ortho-normal eigenstates of $J^2$ and $J_z$ with $j = 1$ and $m_z = m$.

Also, note that $R\vec{\epsilon}(m) = \sum_{m'} D^{(1)}_{mm'}(R)\vec{\epsilon}(m')$.

# Relativistic Spin-1 States

We now want to construct polarization 4-vectors (for massive particles).

- We need to reduce the number of independent components, the only measured quantity is $p^\mu$, the constraint equation can be written as, $p^\mu \epsilon_\mu(p,m) = 0$.
- In the particle's rest frame, $p^\mu \epsilon_\mu = 0$ forces the energy component to zero. Thus, the rest frame states are $\epsilon_\mu(0,m) = (0, \vec{\epsilon}(m))$.
- To get the polarization 4-vector in any frame, simply boost it, $\epsilon^\mu(p,m) = \Lambda^\mu{}_\nu \epsilon^\nu(0,m)$

# Relativistic Spin-1 States

- We can verify that $\epsilon^\mu(p,m)$ has the correct rotational property,

$$\begin{aligned} R^\mu{}_\nu \epsilon^\nu(p,m) &= R^\mu{}_\rho \Lambda^\rho{}_\pi(p)\epsilon^\pi(p_{rf},m) \\ &= R^\mu{}_\rho \Lambda^\rho{}_\pi(p)(R^{-1})^\pi{}_\delta R^\delta{}_\sigma \epsilon^\sigma(p_{rf},m) \\ &= \Lambda^\mu{}_\delta(Rp)\sum_{m'} D^{(1)}_{m'm}(R)\epsilon^\delta(p_{rf},m') \\ &= \sum_{m'} D^{(1)}_{m'm}(R)\epsilon^\mu(Rp,m'), \end{aligned}$$

- We also define the spin-1 projection operator as,

$$P^{(1)}_{\mu\nu}(p) = \sum_m \epsilon_\mu(p,m)\epsilon^*_\nu(p,m) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \equiv -\tilde{g}_{\mu\nu}.$$

# Relativistic Spin-J States

The spin-J states are constructed out of the spin-1 states according to,

$$\epsilon_{\mu_1\mu_2\ldots\mu_J}(p,m) = \sum_{m_{J-1},m_1} ((J-1)m_{J-1}1m_1|Jm) \times \epsilon_{\mu_1\mu_2\ldots\mu_{J-1}}(p,m_{J-1})\epsilon_{\mu_J}(p,m_1).$$

- Rarita-Schwinger conditions for spin-J reduce the number of independent elements from $4^J$ to $(2J+1)$,

$$\begin{aligned} p^{\mu_i}\epsilon_{\mu_1\mu_2\ldots\mu_i\ldots\mu_J}(p,m) &= 0 \\ \epsilon_{\mu_1\mu_2\ldots\mu_i\ldots\mu_j\ldots\mu_J}(p,m) &= \epsilon_{\mu_1\mu_2\ldots\mu_j\ldots\mu_i\ldots\mu_J}(p,m) \\ g^{\mu_i\mu_j}\epsilon_{\mu_1\mu_2\ldots\mu_i\ldots\mu_j\ldots\mu_J}(p,m) &= 0 \end{aligned}$$

- The spin-J projection operator is,

$$P^{(J)}_{\mu_1\mu_2\ldots\mu_J\nu_1\nu_2\ldots\nu_J}(p) = \sum_m \epsilon_{\mu_1\mu_2\ldots\mu_J}(p,m)\epsilon^*_{\nu_1\nu_2\ldots\nu_J}(p,m)$$

# General Technique

- Construct the relativistic spin-1/2 states using Dirac formalism
- Couple these to the spin-J states from the previous section to build higher half-integral spin states.

# Relativistic Spin-1/2 States

Dirac formalism uses a 4-component spinor, $u(p, m)$, for spin-1/2 particles.

- The Dirac equation, $(\gamma^\mu p_\mu - w)u(p, m) = 0$, reduces the number of independent elements to 2.
- In the particle's rest frame, the spinor is $u(0, m) = \begin{pmatrix} \chi(m) \\ 0 \end{pmatrix}$.
- To get the spinor in any frame, boost it,

$$u(p, m) = \Lambda_{\frac{1}{2}}(p)u(0, m) = \sqrt{\frac{E + w}{2w}} \begin{pmatrix} \chi(m) \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+w}\chi(m) \end{pmatrix},$$

- The spin-1/2 projection operator is,
$P^{(\frac{1}{2})}(p) = \sum_m u(p, m)\bar{u}(p, m) = \frac{1}{2w}(\gamma^\mu p_\mu + w)$, where $\bar{u} \equiv u^\dagger \gamma^0$.

# Useful Transformation Properties

Useful properties,

- $u \rightarrow \Lambda_{\frac{1}{2}} u$
- $\bar{u} \rightarrow \bar{u}\Lambda_{\frac{1}{2}}^{-1}$
- $\Lambda_{\frac{1}{2}}^{-1}\gamma^{\mu}\Lambda_{\frac{1}{2}} = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$

Transformation properties of $\bar{u}\Gamma u$,

- $\Gamma = 1$ *scalar*
- $\Gamma = \gamma^5$ *pseudo-scalar*
- $\Gamma = \gamma^{\mu}$ *vector*
- $\Gamma = \gamma^{\mu}\gamma^5$ *pseudo-vector*
- $\Gamma = \gamma^{\mu}\gamma^{\nu}$ *tensor*

# Relativistic Spin-J/2 States

The spin-J/2 states are constructed out of the spin-1/2 and spin-n, where J/2 = n + 1/2, states,

$$u_{\mu_1\mu_2......\mu_n}(p,m) = \sum_{m_n m_{\frac{1}{2}}} (n m_n \frac{1}{2} m_{\frac{1}{2}} | Jm) \epsilon_{\mu_1\mu_2......\mu_n}(p,m_n) u(p,m_{\frac{1}{2}}),$$

- The Rarita-Schwinger conditions for spin-J/2 are,

$$\begin{aligned}
(\gamma^\mu p_\mu - w) u_{\mu_1\mu_2......\mu_n}(p,m) &= 0 \\
u_{\mu_1\mu_2...\mu_i...\mu_j...\mu_n}(p,m) &= u_{\mu_1\mu_2...\mu_j...\mu_i...\mu_n}(p,m) \\
p^{\mu_i} u_{\mu_1\mu_2...\mu_i...\mu_n}(p,m) &= 0 \\
\gamma^{\mu_i} u_{\mu_1\mu_2...\mu_i...\mu_n}(p,m) &= 0 \\
g^{\mu_i\mu_j} u_{\mu_1\mu_2...\mu_i...\mu_j...\mu_n}(p,m) &= 0
\end{aligned}$$

- The projection operator is,
$$P^{(J/2)}_{\mu_1\mu_2...\mu_n\nu_1\nu_2...\nu_n}(p) = \sum_m u_{\mu_1\mu_2...\mu_n}(p,m)\bar{u}_{\nu_1\nu_2...\nu_n}(p,m)$$

# Relativistic Orbital Angular Momentum

Consider $X \to a + b$, with daughter momenta $p_a, p_b$, total momentum $P = p_a + p_b$ and relative momentum $p_{ab} = \frac{1}{2}(p_a - p_b)$.

- Construct a state of pure angular momentum, $\ell$,

$$L^{(\ell)}_{\mu_1\mu_2\ldots\mu_\ell}(p_{ab}) = (-)^\ell P^{(\ell)}_{\mu_1\mu_2\ldots\mu_\ell\nu_1\nu_2\ldots\nu_\ell}(P) p^{\nu_1}_{ab} p^{\nu_2}_{ab} \ldots p^{\nu_\ell}_{ab}.$$

- $L^{(\ell)}_{\mu_1\mu_2\ldots\mu_\ell}$ satisfies the Rarita-Schwinger conditions (it has $(2\ell + 1)$ independent elements),

$$\begin{aligned}
P^{\mu_i} L^{(\ell)}_{\mu_1\mu_2\ldots\mu_i\ldots\mu_\ell}(p_{ab}) &= 0 \\
L^{(\ell)}_{\mu_1\mu_2\ldots\mu_i\ldots\mu_j\ldots\mu_\ell}(p_{ab}) &= L^{(\ell)}_{\mu_1\mu_2\ldots\mu_j\ldots\mu_i\ldots\mu_\ell}(p_{ab}) \\
g^{\mu_i\mu_j} L^{(\ell)}_{\mu_1\mu_2\ldots\mu_i\ldots\mu_j\ldots\mu_\ell}(p_{ab}) &= 0,
\end{aligned}$$

# Orbital Tensor Example

Let's look at this for $\ell = 2$.

- $L^{(2)}_{\mu_1\mu_2}(p_{ab}) = \tilde{p}^{ab}_{\mu_1}\tilde{p}^{ab}_{\mu_2} - \frac{1}{3}\tilde{p}^2_{ab}\tilde{g}_{\mu_1\mu_2}$, where $\tilde{g}_{\mu_1\mu_2} = g_{\mu_1\mu_2} - \frac{P_{\mu_1}P_{\mu_2}}{P^2}$ and $\tilde{p}^{ab}_{\mu} = \tilde{g}_{\mu\nu}p^{\nu}_{ab}$.
- In the *CM* frame, $\tilde{p}^{ab}_{\mu} \to \vec{p}_a = \vec{p}$ and $L^{(2)}_{\mu_1\mu_2} \to L^{(2)}_{ij} = p_i p_j - \frac{1}{3}p^2\delta_{ij}$.
- Project out an $M_z$ piece,

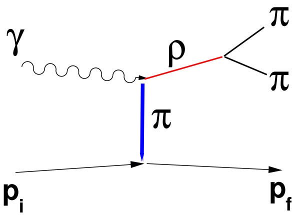
$$
\begin{aligned}
\langle L = 2 | M_z = 1\rangle &\to \langle L = 2 | m_1 = 1; m_2 = 0\rangle \\
&\to L^{(2)}_{ij}\epsilon^i(1)\epsilon^j(0) \\
&= (\vec{p}\cdot\vec{\epsilon}(1))(\vec{p}\cdot\vec{\epsilon}(0)) - \frac{1}{3}p^2\vec{\epsilon}(1)\cdot\vec{\epsilon}(0) \\
&= p^2 cos(\theta) sin(\theta) e^{i\phi} \\
&\sim p^2 Y_{21}(\theta,\phi)
\end{aligned}
$$

# Rules for Writing PWA Amplitudes

- Amplitude must be a Lorentz Scalar, gauge invariant (current conserving) and, where applicable, conserve parity.
- initial (final) spin-0 particles $\rightarrow 1$
- in ital (final) spin-1/2 particles $\rightarrow u(p,m)\,(\bar{u}(p,m))$
- initial (final) spin-1 particles $\rightarrow \epsilon^{\mu}(p,m)\,(\epsilon^{*\mu}(p,m))$
- intermediate spin-j particles $\rightarrow P^{(j)}(p)$
- If a pure orbital angular momentum amplitude is needed, then *all* factors of $p_a, p_b$ must in $L^{(\ell)}_{\mu_1\mu_2\ldots\mu_\ell}(p_{ab})$.
- The only other allowed components are $g_{\mu\nu}$, $\epsilon_{\mu\nu\rho\sigma}$,$\gamma^{\mu}$,$\gamma^5$, factors of momentum (if not looking at a specific orbital state) and any Lorentz scalars (complex constants, $p^2$, . . . ).

# $\gamma p \to p\rho \to p\pi\pi$ $t-$channel

A simple example that could apply to Gluex, $\rho$ photo-production off the proton. In this example, we'll consider only $\pi$ exchange in $t-$channel.




- First consider $\rho \to \pi\pi$, which proceeds only with $\ell = 1$. The amplitude can be written as,

$$L^{(1)}_{\mu}(p_{\pi\pi})\epsilon^{\mu}(p_{\rho}, m_{\rho})$$

# $\gamma p \to p\rho \to p\pi\pi$ $t-$channel

- The $\gamma\pi\rho$ vertex must look like,

$$\epsilon^{*\mu}(p_\rho, m_\rho) X_{\mu\nu} \epsilon^\nu(p_\gamma, m_\gamma)$$

Parity conservation $\to X_{\mu\nu}$ has *negative* parity, so we write it as,

$$X_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} p_\rho^\alpha p_\gamma^\beta$$

Therefore, the amplitude is written as,

$$\epsilon^{*\mu}(p_\rho, m_\rho) \epsilon_{\mu\nu\alpha\beta} p_\rho^\alpha p_\gamma^\beta \epsilon^\nu(p_\gamma, m_\gamma)$$

# $\gamma p \to p\rho \to p\pi\pi$ $t-$channel

- The $pp\pi$ vertex must look like,

$$\bar{u}(p_f, m_f) X u(p_i, m_i)$$

Parity conservation $\to$ X has *negative* parity, so we write the amplitude as,

$$\bar{u}(p_f, m_f)\gamma^5 u(p_i, m_i)$$

# $\gamma p \to p\rho \to p\pi\pi$ $t-$channel

- Putting this all together, the amplitude is,

$$L^{(1)}_{\sigma}(p_{\pi\pi})\left(\sum_{m_\rho}\epsilon^{\sigma}(p_\rho,m_\rho)\epsilon^{*\mu}(p_\rho,m_\rho)\right)\epsilon_{\mu\nu\alpha\beta}p^{\alpha}_{\rho}p^{\beta}_{\gamma}\epsilon^{\nu}(p_\gamma,m_\gamma)$$
$$\times\bar{u}(p_f,m_f)\gamma^5 u(p_i,m_i)BW(p_\rho)R_\pi(p_t)$$

- Which can be written using the spin-1 projection operator as,

$$L^{(1)}_{\sigma}(p_{\pi\pi})P^{(1)\sigma\mu}(p_\rho)\epsilon_{\mu\nu\alpha\beta}p^{\alpha}_{\rho}p^{\beta}_{\gamma}\epsilon^{\nu}(p_\gamma,m_\gamma)$$
$$\times\bar{u}(p_f,m_f)\gamma^5 u(p_i,m_i)BW(p_\rho)R_\pi(p_t)$$

# $\gamma p \to p\rho \to p\pi\pi$ $t-$channel Code

How do we code up this example?

- The amplitude,

$$L_\sigma^{(1)}(p_{\pi\pi})P^{(1)\sigma\mu}(p_\rho)\epsilon_{\mu\nu\alpha\beta}p_\rho^\alpha p_\gamma^\beta \epsilon^\nu(p_\gamma, m_\gamma) \\ \times \bar{u}(p_f, m_f)\gamma^5 u(p_i, m_i) BW(p_\rho) R_\pi(p_t)$$

- The code,

```
L1*rho.Projector()*(levi_civita|(p4_rho%p4_gamma%eps(m_gamma)))
Bar(uf(m_f))*gamma5*ui(m_i)*BreightWigner(p4_rho,mass_rho,width_rho)
*ReggePropagator(t,s,0,pi_slope,pi_inter,pi_sig)
```

# Summary

- Tensor formalism is Relativistic
- Tensor formalism handles $s-, t-, u-$channel in a natural way
- PWA parameters $\longrightarrow$ physics couplings
- We've developed code to take care of the difficulties...so why not use it?