Deflections of Cathode Planes Due to Electrostatic Pressure

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Abstract

When the anode wires are at high voltage, electrostatic pressure is exerted on the cathode planes, which causes the planes to bow in towards the wires near the center of the chamber unless they are supported by rigid backing material. This note describes the formalism for estimating this effect assuming that each cathode can be treated as continuous conducting membrane and that the number of wires in the chamber is large. For a given maximum allowed deflection, I estimate the minimum tension that needs to be applied to the cathode plane before it is attached to its frame.

The performance of an FDC chamber will depend critically on the consistency of the anode-cathode separation h. Variations in h cause variations in gain across the surface of the detector, which can worsen the resolution of positions derived from the cathode strip data and can also impact the dE/dx capabilities of the device. If the cathode planes are unbiased, when the wires have high voltage applied to them the cathode planes will be attracted towards the wire plane due to electrostatic forces. The purpose of this note is to describe this effect in an approximate fashion so that I can estimate the minimum tension I need to apply to the cathode planes during stretching to satisfy a certain deflection tolerance, assuming that there is no additional backing material to stiffen the structure.

I approximate each cathode plane as a circular membrane of thickness t and radius a clamped at the perimeter. Figure 1 provides a sketch of the geometry. The membrane material is assumed to be isotropic (i.e., I ignore

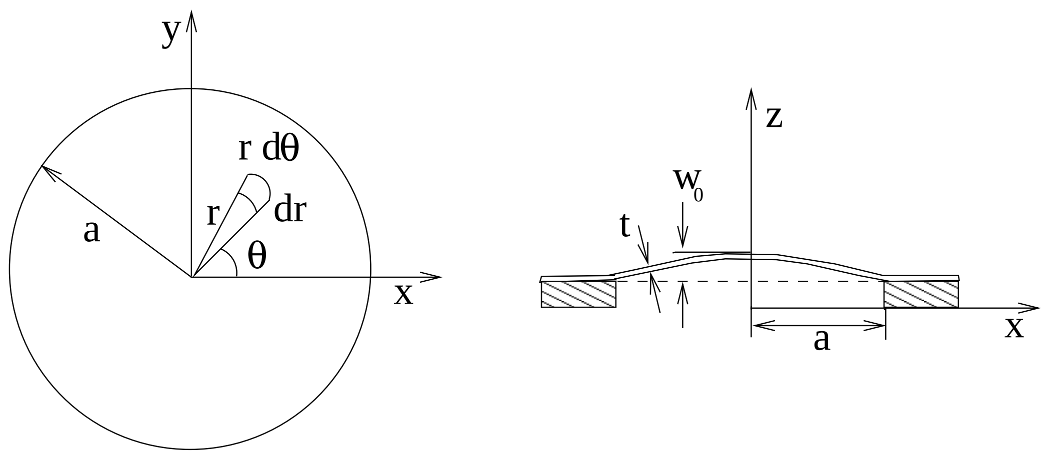

Figure 1: Sketches of membrane geometry. Left: top view. Right: side view.

the gaps between the cathode strips) with Young's modulus E and Poisson's ratio ν. For the purposes of estimating the degree of the bowing effect I assume that the electrostatic pressure p at the surface of the cathode planes is independent of position. The deflection in the direction transverse to the membrane surface will be denoted $w(r)$. The application of the pressure also causes deflections in the radial direction, which I denote $u(r)$. Subject to the boundary conditions $u(a) = 0 = w(a)$, $u(r)$ and $w(r)$ have the following forms:

$$w(r) = w_0 \left(1 - \frac{r^2}{a^2}\right)^2, \quad (1)$$

$$u(r) = r(a-r)(c_0 + c_1 r + c_2 r^2 + \cdots). \quad (2)$$

I truncate the infinite series in $u(r)$ at c_2. The strains in the radial and the tangential directions are given by

$$\epsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2, \quad (3)$$

$$\epsilon_t = \frac{u}{r}. \quad (4)$$

The work done by uniform pressure P applied to the membrane in the direction normal to the undeflected plane's surface is given by

$$W = 2\pi \int_0^a P w(r) r dr. \quad (5)$$

At equilibrium this is matched by the strain energy V_0 due to bending, the strain energy V_1 due to stretching of the mid-plane, and the strain energy V_2 due to the internal tensile stress σ of the membrane, where

$$\begin{aligned} V_0 &= \pi D \int_0^a \left[\left(\frac{d^2 w}{dr^2} \right)^2 + \frac{1}{r^2} \left(\frac{dw}{dr} \right)^2 + \frac{2\nu}{r} \frac{dw}{dr} \frac{d^2 w}{dr^2} \right] r dr, & (6) \\ V_1 &= \frac{\pi E t}{1-\nu^2} \int_0^a (\epsilon_r^2 + \epsilon_t^2 + 2\nu \epsilon_r \epsilon_t) r dr, & (7) \\ V_2 &= 2\pi \sigma t \int_0^a (\epsilon_r + \epsilon_t) r dr. & (8) \end{aligned}$$

I define

$$D = \frac{E t^3}{12(1-\nu^2)}. \quad (9)$$

Plugging in the expressions for w, ϵ_r and ϵ_t, I obtain

$$\begin{aligned} V_0 &= \frac{32\pi}{3} \frac{w_0^2 D}{a^2}, & (10) \\ V_1 &= \frac{\pi E t}{1-\nu^2} \left[\frac{32 w_0^4}{105 a^2} + \frac{k_0^2 a^4}{4} + \frac{3 a^5 k_0 k_1}{10} + \frac{7 a^6 k_1^2}{60} + \frac{a^6 k_0 k_2}{5} + \frac{19 a^7 k_1 k_2}{105} + \frac{13 a^8 k_2^2}{168} \right. \\ &\quad \left. + 2 w_0^2 k_0 a \left(\frac{-23 + 41\nu}{315} \right) + 4 w_0^2 a^2 k_1 \left(\frac{1 + 11\nu}{315} \right) + 4 w_0^2 a^3 k_2 \left(\frac{53 + 71\nu}{3465} \right) \right], & (11) \\ V_2 &= \frac{2\pi}{3} \sigma t w_0^2. & (12) \end{aligned}$$

To obtain the k_i's I require that the total energy of the membrane at equilibrium be a minimum, which means that

$$\frac{\partial V_1}{\partial k_i} = 0. \quad (13)$$

Solving the resulting system of equations, I obtain

$$\begin{aligned} k_0 &= \left(\frac{1585 - 723\nu}{1386} \right) \frac{w_0^2}{a^3}, & (14) \\ k_1 &= \frac{17}{198} (9 + 5\nu) \frac{w_0^2}{a^4}, & (15) \\ k_2 &= \frac{32}{99} (-3 + 2\nu) \frac{w_0^2}{a^5}. & (16) \end{aligned}$$

Then

$$V_1 = \frac{\pi E t}{1-\nu^2}\left(\frac{896585 + 529610\nu - 342831\nu^2}{4802490}\right)\frac{w_0^4}{a^2}. \quad (17)$$

To find w_0, following Timoshenko and Woinosky-Krieger[1] I apply the "principle of virtual displacements":

$$\frac{\partial}{\partial w_0}(V_0 + V_1 + V_2)\,\delta w_0 = 2\pi \int_0^a P\delta w \; r dr \quad (18)$$

$$= 2\pi P \delta w_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right)^2 r dr = \frac{\pi P a^2}{3}\delta w_0, \quad (19)$$

from which I obtain

$$P = 2Et\left(\frac{896585 + 529610\nu - 342831\nu^2}{800415(1-\nu^2)}\right)\frac{w_0^3}{a^4} + \left(4\sigma t + \frac{64D}{a^2}\right)\frac{w_0}{a^2}. \quad (20)$$

Poission's ratio for both Copper and Kapton is about $\nu = 0.34$, so

$$P = 2.93Et\frac{w_0^3}{a^4} + \left(4\sigma t + \frac{64D}{a^2}\right)\frac{w_0}{a^2}. \quad (21)$$

Adding additional terms to the expansion for $u(r)$ does not change the factor in the first term on the right hand side appreciably; Loy et al.[2] truncated the expansion after 5 terms and for $\nu = 0.34$ they get 2.88, within 1.7% of our estimate using 3 terms. To ensure a maximum deflection of $w_{0,max}$, the tension per unit length $T = \sigma t$ applied to the membrane during the initial stretching step should be

$$T \geq \frac{Pa^2}{4w_{0,max}} - \frac{16D}{a^2} - \frac{2.93Etw_{0,max}^2}{4a^2} \approx \frac{Pa^2}{4w_{0,max}}. \quad (22)$$

The approximate form is for small deflections in thin membranes with low flexural rigidity ($D \to 0$) where the internal tensile stress dominates.

In order to estimate the electrostatic pressure, I need to determine the electric field at the surface of each cathode plane. For the purpose of this calculation I ignore the gaps between the cathode strips (i.e., the cathodes be considered to be infinite conducting planes at zero potential; see figure 2). The wires are assumed to be oriented along the z-axis at $y = 0$ and evenly

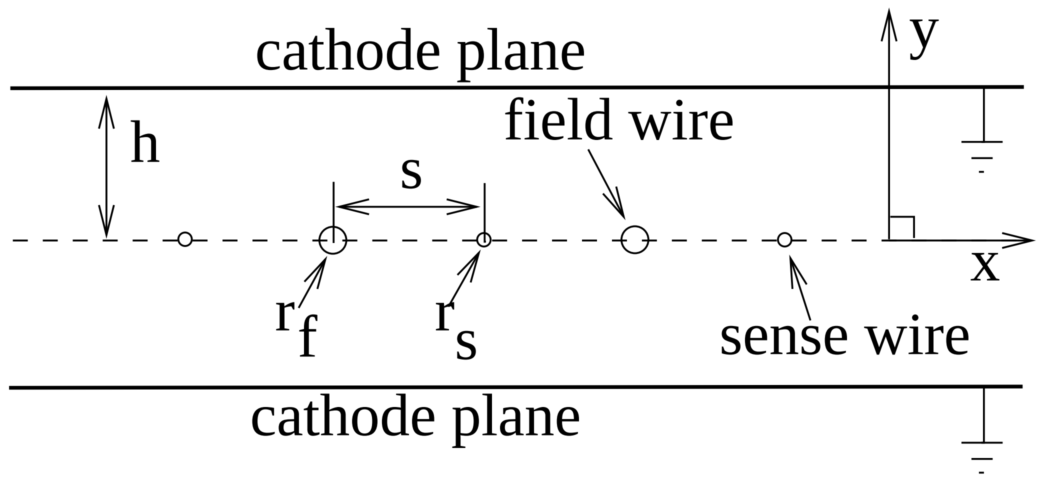

Figure 2: Sketch of the geometry of a multiwire chamber.

separated by a distance s. The wire plane–cathode plane separation is h. Following Mathieson[3], the potential due to the wires is given by

$$P_w(x,y) = -\frac{1}{2\pi\varepsilon_0}\sum_n C_n V_n \ln\left[\frac{\cosh\frac{\pi(x-ns)}{2h} - \cos\frac{\pi y}{2h}}{\cosh\frac{\pi(x-ns)}{2h} + \cos\frac{\pi y}{2h}}\right]^{1/2}, \tag{23}$$

where V_n is the voltage applied to wire n and C_n is the capacitance per unit length of wire n. The electric field vector is determined by taking partial derivatives of $P_w(x,y)$ with respect to x and y:

$$\begin{aligned} E_x(x,y) &= -\frac{\partial P_w(x,y)}{\partial x} \\ &= \frac{1}{4h\varepsilon_0}\sum_n \frac{C_n V_n \sinh\frac{\pi(x-ns)}{2h}\cos\frac{\pi y}{2h}}{(\cosh\frac{\pi(x-ns)}{2h} - \cos\frac{\pi y}{2h})(\cosh\frac{\pi(x-ns)}{2h} + \cos\frac{\pi y}{2h})} \quad (24) \\ E_y(x,y) &= -\frac{\partial P_w(x,y)}{\partial y} \\ &= \frac{1}{4h\varepsilon_0}\sum_n \frac{C_n V_n \cosh\frac{\pi(x-ns)}{2h}\sin\frac{\pi y}{2h}}{(\cosh\frac{\pi(x-ns)}{2h} - \cos\frac{\pi y}{2h})(\cosh\frac{\pi(x-ns)}{2h} + \cos\frac{\pi y}{2h})} \quad (25) \end{aligned}$$

At $y = h$, the field component in the x-direction vanishes ($E_x(x,h) = 0$) and $E_y(x,h)$ reduces to

$$E_y(x,h) = \frac{1}{4h\varepsilon_0}\sum_n C_n V_n \,\mathrm{sech}\frac{\pi(x-ns)}{2h}. \tag{26}$$

Suppose I have a large number of field wires (radius r_f) at potential V_f and positioned at odd values of n and large number of sense wires (radius r_s) at potential V_s and positioned at even values of n and that the dimensions of the chamber are much larger than h. Then near the center of the chamber I can assume that the C_n's can be appoximated by two constants, C_s and C_f, and the electric field at $y = h$ becomes

$$E_y(x,h) = \frac{1}{4h\varepsilon_0}\left[C_s V_s \sum_{n\ even} \operatorname{sech}\frac{\pi(x-ns)}{2h} + C_f V_f \sum_{n\ odd} \operatorname{sech}\frac{\pi(x-ns)}{2h}\right]. \tag{27}$$

For $n \to \infty$, I approximate the sums with integrals:

$$\begin{aligned}\sum_{n\ even} \operatorname{sech}\frac{\pi(x-ns)}{2h} &= \sum_m \operatorname{sech}\frac{\pi(x-2ms)}{2h} \approx \int_{-\infty}^{\infty} \operatorname{sech}\frac{\pi(x-2ms)}{2h} dm \\ &= -\frac{2h}{\pi s}\tan^{-1}\exp\frac{\pi(x-2ms)}{2h}\Big|_{-\infty}^{+\infty} = \frac{h}{s}.\end{aligned} \tag{28}$$

Then

$$E_y(x,h) = \frac{C_s V_s + C_f V_f}{4\varepsilon_0 s}. \tag{29}$$

If I replace all of the field wires with sense wires, this expression becomes

$$E_y(x,h) = \frac{C_s V_s}{2\varepsilon_0 s}\ \text{(sense wires only)}, \tag{30}$$

which is equation 41 in Sauli[4]. In this approximation I find that E_y near the surface of the cathode planes is independent of x.

The force on a conductor due to an electric field $\vec{E}$ is given by [5]

$$\vec{F} = \frac{1}{2}\oint_S \vec{E}\sigma dA, \tag{31}$$

integrating over the surface S of the conductor. Here σ is the charge density on the surface. In the limit where $\vec{E} \to$ constant, $\sigma \to$ constant and I can write the electrostatic pressure as

$$p = \frac{dF}{dA} = \frac{1}{2}\varepsilon_0 E_y^2(x,h) = \frac{(C_s V_s + C_f V_f)^2}{32\epsilon_0 s^2}. \tag{32}$$

For the case where there are only sense wires in the system, this becomes

$$p = \frac{C_s^2 V_s^2}{8\varepsilon_0 s^2}, \tag{33}$$

which is Sauli's equation 46[4].

Next I need to find expressions for the capacitances per unit length C_f and C_s. For the sense wire at $\{x, y\} = \{0, 0\}$, the potential on the surface of the wire must be V_s:

$$V_s = -\sum_n k_n L_{n0} = -\left[k_0 \sum_{n=even} L_{n0} + k_1 \sum_{n=odd} L_{n0}\right], \tag{34}$$

where k_0 and k_1 are constants related to the capacitance, and

$$L_{00} = \ln \frac{\pi r_s}{4h}, \tag{35}$$

$$L_{n0} = \ln \left|\tanh \frac{\pi s n}{4h}\right|, \text{ for } n \neq 0. \tag{36}$$

For the field wire at $\{x, y\} = \{s, 0\}$, the potential at the surface of the wire must be V_f:

$$V_f = -\sum_n k_n L_{n1} = -\left[k_0 \sum_{n=even} L_{n1} + k_1 \sum_{n=odd} L_{n1}\right], \tag{37}$$

where

$$L_{11} = \ln \frac{\pi r_f}{4h}, \tag{38}$$

$$L_{n1} = \ln \left|\tanh \frac{\pi s(n-1)}{4h}\right|, \text{ for } n \neq 1. \tag{39}$$

Equations 34 and 37 form a system of two equations in two unknowns. After some algebra,

$$k_0 = \frac{V_f \sum_{odd} L_{n0} - V_s \sum_{odd} L_{n1}}{(\sum_{odd} L_{n1})(\sum_{even} L_{n0}) - (\sum_{odd} L_{n0})(\sum_{even} L_{n1})}, \tag{40}$$

$$k_1 = \frac{V_s \sum_{even} L_{n1} - V_f \sum_{even} L_{n0}}{(\sum_{odd} L_{n1})(\sum_{even} L_{n0}) - (\sum_{odd} L_{n0})(\sum_{even} L_{n1})}, \tag{41}$$

where the sums have the following forms:

$$\sum_{odd} L_{n0} = \sum_{even} L_{n1} = 2 \sum_{m=0}^{\infty} \ln \tanh \frac{\pi s(2m+1)}{4h}, \tag{42}$$

$$\sum_{even} L_{n0} = \ln \frac{\pi r_s}{4h} + 2 \sum_{m=1}^{\infty} \ln \tanh \frac{\pi s m}{2h}, \tag{43}$$

$$\sum_{odd} L_{n1} = \ln \frac{\pi r_f}{4h} + 2 \sum_{m=1}^{\infty} \ln \tanh \frac{\pi s m}{2h}. \tag{44}$$

The constants k_0 and k_1 are related to the capacitance per unit length according to $C_s = 2\pi\varepsilon_0 k_0$ and $C_f = 2\pi\varepsilon_0 k_1$ at unit potential. The nominal detector geometry calls for $s = h$, for which the expressions for C_s and C_f simplify to

$$C_s = \frac{2\pi\varepsilon_0 \left(\ln \frac{\pi r_f}{4h} - 0.18 + 0.88\frac{V_f}{V_s}\right)}{0.74 + 0.18(\ln \frac{\pi r_s}{4h} + \frac{\pi r_f}{4h}) - \ln \frac{\pi r_f}{4h} \ln \frac{\pi r_s}{4h}}, \tag{45}$$

$$C_f = \frac{2\pi\varepsilon_0 \left(\ln \frac{\pi r_s}{4h} - 0.18 + 0.88\frac{V_s}{V_f}\right)}{0.74 + 0.18(\ln \frac{\pi r_s}{4h} + \frac{\pi r_f}{4h}) - \ln \frac{\pi r_f}{4h} \ln \frac{\pi r_s}{4h}}, \tag{46}$$

provided that $V_f \neq 0$. For the case where the field wires are replaced by sense wires, the capacitance per unit length of the wires is

$$C_s = \frac{2\pi\varepsilon_0}{1.06 - \ln \frac{\pi r_s}{4h}}, \tag{47}$$

which is numerically equivalent to Sauli's equation 39 when $s = h$. Combining equations 32, 45 and 46, I get

$$p = \frac{\pi^2\varepsilon_0}{8s^2}\left(\frac{V_s \ln \frac{\pi r_f}{4h} + V_f \ln \frac{\pi r_s}{4h} + 0.70(V_s + V_f)}{0.74 + 0.18(\ln \frac{\pi r_s}{4h} + \frac{\pi r_f}{4h}) - \ln \frac{\pi r_f}{4h} \ln \frac{\pi r_s}{4h}}\right)^2. \tag{48}$$

Using $r_s = 10$ μm, $r_f = 40$ μm, $s = h = 5$ mm, $V_s = 2.42$ kV, and $V_f = -0.5$ kV, I get $p = 0.022$ Pa. Thus for a maximum deflection of for example 10 μm for a 1 m diameter membrane, the minimum tension should be 137 N/m, using equation 22, with the caveat that 10 microns is not very small compared to the cathode thickness of 25 microns...

References

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