General Properties Of Three-body Decays

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Abstract

This document derives some general properties of Dalitz plots. In particular, a derivation of the relation between the density along a decay band and the cosine of the decay angles is presented. It is also shown that given two identical particles, how the helicity angles of the two decay chains are related.

1 Introduction

In this note, we are going to examine the properties a scattering process where we have three particles in the final state. We describe this in terms of a projectile particle, P, with mass m_p, interacting with a target particle, T, with mass m_t. This ultimately goes to a final state with three particles, 1, 2 and 3, with masses m_1, m_2 and m_3, respectively.

$$P + T \quad \rightarrow \quad 1 + 2 + 3$$

Throughout the discussion, we will consider that the reaction proceeds through an intermediate particle, A, with mass m_a, and that A decays to 1 and 2.

$$P + T \quad \rightarrow \quad 1 + A \ \rightarrow \ 1 + 2 + 3 \tag{1}$$

In the next section, we describe that various coordinate systems used to describe the reactions. While the reaction takes place in the "lab frame", where the target is initially at rest (Section 2.1, we will find the "center-of-mass frame" (Section 2.2) to be more useful. We will also look at the decay of A in the rest frame of A. Here, we will find that there are three frames that are useful, all of which differ by the orientation of the z axis (Section 2.3).

In Section 3, we look in detail at the Dalitz plot that can be formed from the invariant masses of pairs of the final-state particles. The Dalitz plot is useful for understanding details of the production and decay of the particle A.

2 Coordinate Systems and Kinematics

2.1 The Lab Frame

We consider the reaction in which a projectile particle of mass m_p and initial momentum, $\vec{P}_p$, interacts with a stationary target particle of mass, m_t. After the interaction, two particles emerge. One of mass m_a and momentum $\vec{P}_a$, and the second of mass m_1 and momentum $\vec{P}_1$. The particle of mass m_a subsequently decays into two daughter particles of masses m_2 and m_3, and momentum $\vec{P}_2$ and $\vec{P}_3$, respectively. This is shown in Figure 1(a), and this coordinate system in known as the "lab frame". In this frame, momentum conservation tells us

$$\vec{P}_p \quad = \quad \vec{P}_1 + \vec{P}_2 + \vec{P}_3 \,.$$

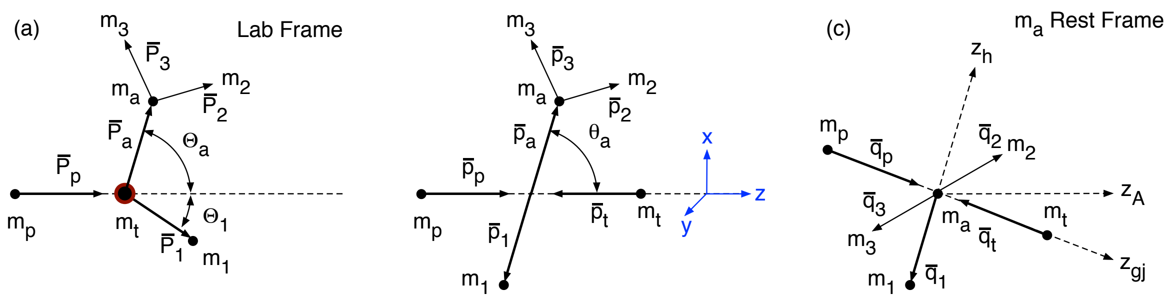

Figure 1: The coordinate

The energy of the particles are given in terms of their momentum and mass as

$$
\begin{aligned}
E_{pL} &= \sqrt{(P_p c)^2 + (m_p c^2)^2} \quad \text{projectile} \\
E_{tL} &= m_t c^2 \quad \text{target} \\
E_{1L} &= \sqrt{(P_1 c)^2 + (m_1 c^2)^2} \\
E_{aL} &= \sqrt{(P_a c)^2 + (m_a c^2)^2} \\
E_{2L} &= \sqrt{(P_2 c)^2 + (m_2 c^2)^2} \\
E_{3L} &= \sqrt{(P_3 c)^2 + (m_3 c^2)^2} \, .
\end{aligned}
$$

Using these, energy conservation gives us

$$E_{pL} + E_{tL} = E_{1L} + E_{2L} + E_{3L} \, .$$

In the lab frame, we have the Mandelstam variables, s and t, given as

$$
\begin{aligned}
s &= \left[E_{pL} + (m_t\, c^2)\right]^2 - (P_p\, c)^2 \\
t &= \left[E_{pL} - E_{aL}\right]^2 - \left(\vec{P}_p - \vec{P}_a\right) \cdot \left(\vec{P}_p - \vec{P}_a\right) c^2 \\
u &= \left[E_{pL} - E_{1L}\right]^2 - \left(\vec{P}_p - \vec{P}_1\right) \cdot \left(\vec{P}_p - \vec{P}_1\right) c^2 \, .
\end{aligned}
$$

2.2 The Center-of-mass Frame

It is also useful to consider the interaction in the center-of-mass frame, as shown in Figure 1(b). In this frame, we use lower case $\vec{p}_j$s to denote the momentum, and E_j to denote the energy. In this frame, the projectile and target particles moving towards each other with equal an opposite momentum

$$0 = \vec{p}_p + \vec{p}_t \, .$$

Particles A and 1 emerge from the interaction with equal and opposite momentum as well, but oriented in space differently than the projectile and target.

$$0 = \vec{p}_1 + \vec{p}_a$$

We can account for this orientation by noting that particle A emerges at an angle θ_a relative to the direction of the projectile particle, P, as shown. Particle A then decays into its two daughter particles, 2 and 3, with momenta $\vec{p}_2$ and $\vec{p}_3$, respectively. The total energy in the center of mass frame, E, is equal to $\sqrt{s}$. The

Lorentz-invariant s, t and u can be written in terms of the center-of-mass momentum and energy

$$\begin{aligned} s &= [E_p + E_t]^2 \\ t &= [E_p - E_a]^2 - (\vec{p}_p - \vec{p}_a) \cdot (\vec{p}_p - \vec{p}_a)\ c^2 \\ u &= [E_p - E_1]^2 - (\vec{p}_p - \vec{p}_1) \cdot (\vec{p}_p - \vec{p}_1)\ c^2 \,. \end{aligned}$$

With total energy E, and $|\vec{p}_a| = |\vec{p}_1| \equiv p_a$, we can write

$$E = \sqrt{(p_a c)^2 + (m_a c^2)^2} + \sqrt{(p_a c)^2 + (m_1 c^2)^2}\,,$$

which can be solved to yield

$$p_a = \frac{\sqrt{[E^2 - (m_1 c^2 + m_a c^2)^2]\ [E^2 - (m_1 c^2 - m_a c^2)^2]}}{2E}\,. \tag{2}$$

In order to change reference frames, we need to perform a Lorentz boost along the direction of the projectile. This transformation is given as the matrix multiplication

$$\begin{pmatrix} E_i + E_t \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E_{iL} + m_t c^2 \\ P_i \end{pmatrix} .M_{ij} \tag{3}$$

Using the second (momentum) equation, we find

$$\beta = \frac{P_p c}{E_{pL} + m_t c^2}\,. \tag{4}$$

The relativistic factor, γ, can be expressed in terms of β as

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}\,.$$

So, we can write

$$\gamma = \frac{E_{pL} + m_t c^2}{\sqrt{(m_t c^2)^2 + 2 m_t c^2 E_{pL}}}\,, \tag{5}$$

and the combination of $\beta\gamma$ is

$$\beta\gamma = \frac{P_p\, c}{\sqrt{(m_t c^2)^2 + 2 m_t c^2 E_{pL}}}\,. \tag{6}$$

In the center-of-mass frame, we choose the z axis to be along the direction of the projectile, $\hat{z} = \hat{p}_p$. With this choice for the z-axis, there is one natural choice for the direction of the y-axis, normal to the reaction plane, defined as

$$\hat{y} = \frac{\vec{p}_p \times \vec{p}_a}{|\vec{p}_p \times \vec{p}_a|}\,,$$

and the x axis is defined by

$$\hat{x} = \hat{y} \times \hat{z}\,.$$

Figure 2(a) shows these choices on the reaction as seen in the center-of-mass frame. This choice is useful when studying the primary interaction, as it keeps all the particles from this reaction in the same plane.

A second choice is to choose one of the coordinate axes normal to the plane that contains the three final-state particles, 1, 2 and 3, and the other two axes in the plane. For this choice, we generally cannot choose the z-axis along the direction of the projectile. Thus, it is convenient to choose it along the direction of particle A,

$$\hat{z} \;=\; \hat{p}_a \,,$$

and the the y-axis is taken normal to the decay plane,

$$\hat{y} \;=\; \frac{\vec{p}_1 \times \vec{p}_2}{|\vec{p}_1 \times \vec{p}_2|} \,.$$

The x-axis is then chosen to yield a right-handed coordinate system, with $\hat{x}$ given by

$$\hat{x} \;=\; \hat{y} \times \hat{z} \,.$$

This choice is shown in Figure 2(b), and is useful when we are studying properties of the final-state particles.

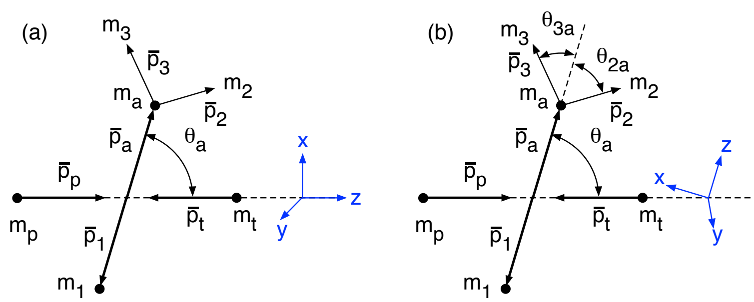

Figure 2: Coordinate systems in the center-of-mass frame. In (a), we have taken the z-axis along the projectile direction, and the y-axis normal to the reaction plane, defined by $\vec{p}_p \times \vec{p}_a$. In (b), we have taken the z-axis along particle A, and the y-axis normal to the decay plane of the three final-state particles, defined by $\vec{p}_1 \times \vec{p}_2$.

In the frame with the z-axis along the projectile direction, the initial two particles have momenta given as

$$\begin{aligned} \vec{p}_p &= p_p \,\hat{z} \\ \vec{p}_t &= -p_p \,\hat{z} \,. \end{aligned}$$

The two secondaries have momentum given as

$$\begin{aligned} \vec{p}_a &= p_a \left(\sin\theta_a \,\hat{x} + \cos\theta_a \,\hat{z}\right) \\ \vec{p}_1 &= -p_a \left(\sin\theta_a \,\hat{x} + \cos\theta_a \,\hat{z}\right) \,. \end{aligned}$$

The two daughters particles, from the decay of the particle A, are not required to be in the reaction place, so we need to define a pair of polar angles to describe each of these. Thus, we have the momentum of particles 2 and 3 given by

$$\begin{aligned} \vec{p}_2 &= p_2 \left(\sin\theta_2 \cos\phi_2 \,\hat{x} + \sin\theta_2 \sin\phi_2 \,\hat{y} + \cos\theta_2 \,\hat{z}\right) \\ \vec{p}_3 &= p_3 \left(\sin\theta_3 \cos\phi_3 \,\hat{x} + \sin\theta_3 \sin\phi_3 \,\hat{y} + + \cos\theta_3 \,\hat{z}\right) \,. \end{aligned}$$

In the choice with the z-axis along the direction of A, we have

$$\begin{aligned} \vec{p}_1 &= -p_a \hat{z} \\ \vec{p}_2 &= p_2 \,\left(\cos\theta_{2a} \hat{z} + \sin\theta_{2a} \hat{x}\right) \\ \vec{p}_3 &= p_3 \,\left(\cos\theta_{3a} \hat{z} + \sin\theta_{3a} \hat{x}\right) \,. \end{aligned}$$

2.3 The Rest Frame of the Decaying Particle

The particle of mass m_a subsequently decays into daughters of masses m_2 and m_3. It is also convient to examine the system as viewed from the rest frame of the particle of mass m_a. We start with the Lorentz boost that goes from the overall center-of-mass frame to this frame. It is a boost along the direction of the particle, and taking the lead from the previos section, we have that

$$\beta = \frac{p_a c}{\sqrt{(p_a c)^2 + (m_a c^2)^2}}$$

where p_a is given by equation 2. Similarly,

$$\gamma = \frac{\sqrt{(p_a c)^2 + (m_a c^2)^2}}{m_a c^2},$$

and

$$\beta\gamma = \frac{p_a c}{m_a c^2}.$$

There are three coordinate systems that are used to describe the decay of particle with mass m_a. These differ by the choice of the direction of the z axis. These are the "helicity frame", the "Gottfried-Jackson frame", and the the "Adair frame". For all three of these, the y axis is chosen normal to the reaction plane:

$$\hat{y} = \frac{\vec{p}_i \times \vec{p}_a}{|\vec{p}_i \times \vec{p}_a|},$$

and the x axis is chosen according to

$$\hat{x} = \hat{y} \times \hat{z}.$$

Figure 3 shows the z axis for the three coordinate systems.

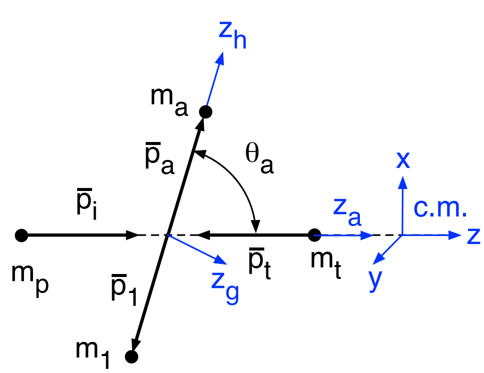

Figure 3: The z-axis for the helicity (z_h), Adair (z_a) and the Gottfried-Jackson (z_g) frames. Also shown are the coordinate axes for the center-of-mass frame. The y axis is the same in all four frames, the x axis is determined by $\hat{x} = \hat{y} \times \hat{z}$.

2.3.1 Helicity Frame

For the helicity frame, the z axis is taken along the the direction of the particle of mass m_a as seen in the center-of-mass frame. This is also along the boost direction which moves from the center-of-mass frame to the rest frame of particle m_a. Given the coice of the coordinate directions, we need to angles to describe the momentum of the final-state particles, θ_h and ϕ_{hel}.

$$\begin{aligned}\vec{p}_2 &= p_a\left(\sin\theta_h\cos\phi_h\hat{x}+\sin\theta_h\sin\phi_h\hat{y}+\cos\theta_h\hat{z}\right)\\ \vec{p}_3 &= -p_a\left(\sin\theta_h\cos\phi_h\hat{x}+\sin\theta_h\sin\phi_h\hat{y}+\cos\theta_h\hat{z}\right)\,.\end{aligned}$$

2.3.2 Adair Frame

In the Adair frame, the the z axis is chosen to be along the initial beam direction in the center-of-mass.

2.3.3 Gottfried-Jackson Frame

In the Gottfried-Jackson frame, the the z axis is chosen to be along the initial beam direction as seen in the rest frame of the particle of mass m_a.

3 Dalitz Plots

In the case where one of the daughters of particle a decays into a pair of daughters as well, for example

$$a \;\rightarrow\; 1+x \;\rightarrow\; 1+2+3$$

we will often use a Dalitz plot to analyze the three-body decay. The Dalitz plot [2] is a convenient way to view and analyze reactions in which there are three particles in the final state. If the three final-state particles are identical, it is sometimes plotted in terms of the kinetic energy of the particles. In the case when not all the particles are indexical, the Dalitz plot is made using the square of the invariant mass of pairs of particles. We also make a change of notation at this point, and move to units in which $c = 1$. Thus, instead of writing m_1c^2 or p_ic, we will just write m_i or p_i. Thus, the relativistic energy relation will be written as

$$E_i^2 \;=\; p_i^2+m_i^2\,.$$

Finally, while most of the quantities at which we look are Lorentz invariants, it is convenient to examine the final state in the center-of-mass frame, where the total energy is E. We also often see the center of mass system written as a particle of mass M, where $M = E$. In addition, when needed, we choose a system in which one of the coordinate axes is normal to the plane containing the three particles.

3.1 Phase Space Distributions in Three-body Final States

We consider some process with center-of-mass energy E which has a final state with three particles, 1, 2 and 3, whose masses are m_1, m_2 and m_3. Each of the three particles has momentum, $\vec{p}_i$, and energy E_i, where the energy, momentum and mass are related by

$$E_i^2 \;=\; m_i^2+\vec{p}_i\cdot\vec{p}_i \;=\; m_i^2+p_i^2\,.$$

For these three particles, we know that the distribution of momenta populates a two-dimensional phase space, d^2R_3. Here, we want to determine the distribution in the 2–D phase space.

To do this, we start with the full 9-D phase space for a three-body final state, given as

$$d^9R_3 \;=\; \frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}\frac{d^3\vec{p}_3}{2E_3}\,\delta^3\left(\vec{p}_1+\vec{p}_2+\vec{p}_3\right)\delta\left(E_1+E_2+E_3-E\right)\,,$$

where the delta functions impose momentum and energy conservation. The $\delta^3\left(\sum\vec{p}_i\right)$ forces the total momentum to be zero and the $\delta\left(\sum\left[E_i\right]-E\right)$ constrains the total energy to be E. We can immediately integrate over $d^3\vec{p}_3$, with the delta function imposing the constraint that $\vec{p}_3 = -\left(\vec{p}_1+\vec{p}_2\right)$. This integral yields

$$d^6R_3 \;=\; \frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}\frac{1}{2E_3}\,\delta\left(E_1+E_2+E_3-E\right)\,. \tag{7}$$

In this expression, we can write E_3 as

$$\begin{aligned} E_3^2 &= m_3^2 + p_3^2 \\ E_3^2 &= m_3^2 + (\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 + \vec{p}_2) \\ E_3^2 &= m_3^2 + p_1^2 + p_2^2 + 2 p_1 p_2 \cos\theta_{12} \, , \end{aligned} \tag{8}$$

where θ_{12} is the angle between $\vec{p}_1$ and $\vec{p}_2$. We can next integrate over the variable p_2.To do this, we choose the z-axis along the direction of $\vec{p}_1$. This allows us to write

$$d^3\vec{p}_2 = p_2^2 \, dp_2 \, d\left(\cos\theta_{12}\right) \, d\phi_{12} \, .$$

This can be integrated over $d\phi_{12}$ to yield a factor of 2π. We can also differentiate equation 8 for fixed values of p_1 and p_2 to obtain

$$E_3 \, dE_3 = p_1 \, p_2 \, d\left(\cos\theta_{12}\right) \, .$$

Combining these, we can write equation 7 as

$$d^4R_3 = \frac{\pi}{4} \frac{d^3\vec{p}_1}{p_1 E_1} \frac{p_2^2 dp_2}{p_2 E_2} \int_{E_{3min}}^{E_{3max}} dE_3 \, \delta\left(E_1 + E_2 + E_3 - E\right) \, ,$$

where the limits of integration can be written as

$$\begin{aligned} E_{3min} &= \sqrt{m_3^2 + (p_1 - p_2)^2} \\ E_{3min} &= \sqrt{m_3^2 + (p_1 + p_2)^2} \, . \end{aligned}$$

The integral over the delta function yields 1, so we find

$$d^4R_3 = \frac{\pi}{4} \frac{d^3\vec{p}_1}{p_1 E_1} \frac{p_2^2 dp_2}{p_2 E_2} \, . \tag{9}$$

We now expand the $d^3\vec{p}_1$ as

$$d^3\vec{p}_1 = p_1^2 \, dp_1 \, d\left(\cos\theta_1\right) \, d\phi_1 \, ,$$

and the angular integral can be carried out to yield 4π. This lets us write equation 9 as a 2-D phase space

$$d^2R_3 = \pi^2 \frac{p_1 \, dp_1}{E_1} \frac{p_2 \, dp_2}{E_2} \, .$$

If we now note that $E_i \, dE_i = p_1 \, dp_i$, we can write this expression as

$$d^2R_3 = \pi^2 \, dE_1 \, dE_2 \, . \tag{10}$$

The allowed region in the E_1–E_2 space is determined by the limits that

$$E_{3min} \leq E - E_1 - E_2 \leq E_{3max} \, .$$

We can define the smooth curve that bounds the allowed region as

$$F(E_1, E_2, E, m_1, m_2, m_3) = 0 \, .$$

We can find this boundary by solving the equation

$$E - E_1 - E_2 = \sqrt{m_3^2 + (p_1 \pm p_2)^2} \, .$$

Doing the messy algebra yields an expression for the function, F, as

$$\begin{aligned} F &= \left(E^2 + m_1^2 + m_2^2 - m_3^2\right)^2 - 4m_1^2 m_2^2 - 8E\left(E_1^2 E_2 + E_1 E_2^2\right) \\ &\quad +4\left(E^2 + m_2^2\right)E_1^2 + 4\left(E^2 + m_1^2\right)E_2^2 + 4\left(3E^2 + m_1^2 + m_2^2 - m_3^2\right)E_1 E_2 \\ &\quad -4\left(E^2 + m_1^2 + m_2^2 - m_3^2\right)EE_1 - 4\left(E^2 + m_1^2 + m_2^2 - m_3^2\right)EE_2 \end{aligned} \tag{11}$$

Equation 10 tells us that if the three final-state particles follow a phase-space distribution, then a plot of E_2 versus E_1 will be uniformly populated. Since there is nothing special about particles 1 and 2, a plot of E_j versus E_i will be uniformly populated. This can also be written in terms of the kinetic energy, $T_i = E_i - m_i$, which would also be uniformly populated.

3.2 The Dalitz Plot

Rather than expressing phase space in terms of energies, it is more typical to express things in terms of invariant masses, m_{ij}^2, where we can write the invariant mass of two of the three particles as

$$\begin{aligned} m_{ij}^2 &= (E_i + E_j)^2 - (\vec{p}_i + \vec{p}_j)^2 \\ m_{ij}^2 &= m_i^2 + m_j^2 + 2(E_i E_j - \vec{p}_i \cdot \vec{p}_j) \\ m_{ij}^2 &= m_i^2 + m_j^2 + 2(E_i E_j - p_i p_j \cos\theta_{ij}). \end{aligned} \tag{12}$$

There is a global constraint relating the total energy, E, the three daughter masses and the three possible invariant masses in the problem. This is given as

$$E^2 = m_{12}^2 + m_{13}^2 + m_{23}^2 - m_1^2 - m_2^2 - m_3^2. \tag{13}$$

Consider now the addition of m_{12}^2 and m_{13}^2, where we replace $\vec{p}_3 = -\vec{p}_1 - \vec{p}_2$ and $E_3 = E - E_1 - E_2$. It is easily shown that

$$E_1 = \frac{E^2 + m_1^2 - m_{23}^2}{2E} \tag{14}$$

$$E_2 = \frac{E^2 + m_2^2 - m_{13}^2}{2E} \tag{15}$$

$$E_3 = \frac{E^2 + m_3^2 - m_{12}^2}{2E}. \tag{16}$$

This confirms that the original statement on energy and invariant-mass squared being equivalent. We see that

$$\pi^2\, dE_1\, dE_2 = \frac{\pi^2}{4E^2}\, dm_{13}^2\, dm_{23}^2\,,$$

so the phase space can be written in terms of a pair of invariant masses as

$$d^2 R_3 = \frac{\pi^2}{4E^2}\, dm_{ij}^2\, dm_{jk}^2\,. \tag{17}$$

A plot of the allowed phase space for the three-body final states in terms of a pair of squared invariant masses is referred to as a Dalitz plot. We can transform the boundary F, as given by equation 11, to an equivalent expression, G, expressed in terms of m_{ij}^2. This gives us

$$\begin{aligned} G(E, m_1, m_2, m_3, m_{12}^2, m_{23}^2) &= m_{12}^4 m_{23}^2 + m_{12}^2 m_{23}^4 - m_{12}^2 m_{23}^2\left(E^2 + m_1^2 + m_2^2 + m_3^2\right) \\ &\quad +m_{12}^2\left(m_3^2 - m_2^2\right)\left(E^2 - m_1^2\right) + m_{23}^2\left(m_1^2 - m_2^2\right)\left(E^2 - m_3^2\right) \\ &\quad +\left(m_2^2 E^2 - m_1^2 m_3^2\right)\left(E^2 - m_1^2 + m_2^2 - m_3^2\right). \end{aligned} \tag{18}$$

As with the expression for F, setting $G = 0$, and then solving for the allowed values of m_{12}^2 and m_{23}^2 will yield the boundary of the Dalitz plot. Such a plot will be bounded by minimum and maximum values of m_{ij}^2, given as

$$\begin{aligned}\min\left(m_{ij}^2\right) &= \left(m_i + m_j\right)^2 \\ \max\left(m_{ij}^2\right) &= \left(E - m_k\right)^2 .\end{aligned}$$

For the minimum value of m_{ij}^2, particles i and j are parallel, with the same velocity, while particle k is moving opposite to these. For the maximum value of m_{ij}^2, particle k is at rest, and i and j have equal and opposite momenta. The boundary can be found by solving equation 18 for m_{12}^2 in terms of m_{23}^2, and then varying m_{23}^2 from $(m_2 + m_3)^2$ to $(E - m_1)^2$ and finding the two allowed values for m_{12}^2. We can write the quadratic equation for m_{12}^2 as

$$\begin{aligned}0 \;=\; & \left[m_{23}^2\right]\left(m_{12}^2\right)^2 + \left[\left(m_{23}^2\right)^2 + \left(m_3^2 - m_2^2\right)\left(E^2 - m_1^2\right) - m_{23}^2\left(E^2 + m_1^2 + m_2^2 + m_3^2\right)\right]\left(m_{12}^2\right) \\ & + \left[m_{23}^2\left(m_1^2 - m_2^2\right)\left(E^2 - m_3^2\right) + \left(m_2^2 E^2 - m_1^2 m_3^2\right)\left(E^2 - m_1^2 + m_2^2 - m_3^2\right)\right] .\end{aligned}$$

The Dalitz plot shows m_{ij}^2 plotted against m_{jk}^2, as shown in Figure 4. We also note that the third invariant mass, m_{ki}^2, can also be read off this plot as well. It follows the diagonal dashed line in the figure, with the maximum value of m_{ki}^2 on the left side and the minimum value of m_{ki}^2 on the right side of the plot.

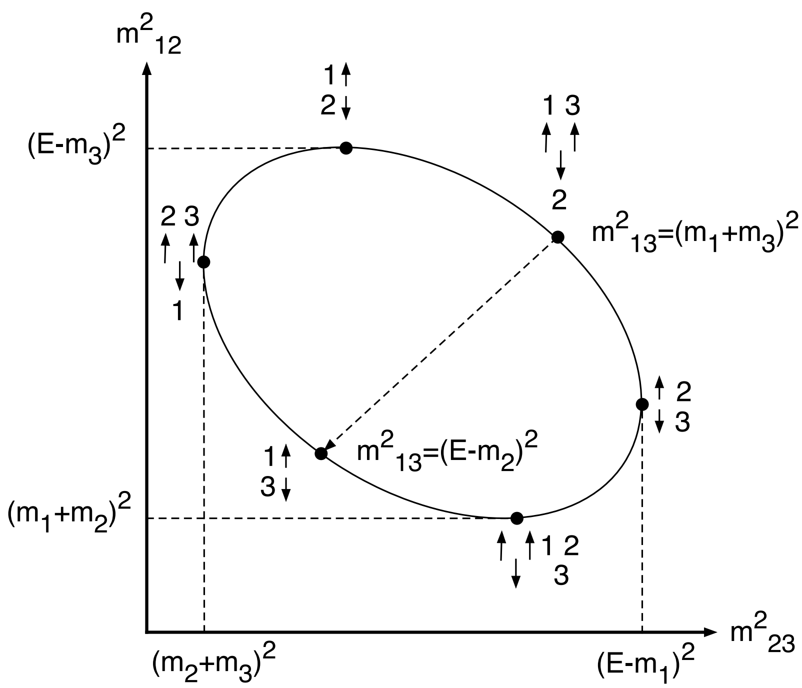

Figure 4: A Dalitz plot for a system with center-of-mass energy E going to a final state with three particles of masses m_1, m_2 and m_3. The allowed phase space is bounded by the curve in the figure. Along the boundary of the figure, all three particles are collinear.

3.3 Distributions in the Dalitz Plot

Now let us consider a situation where our system of total energy E decays initially into to particles, one of which is our particle of mass m_1, and a second which has mass m_a. The particle of mass m_a will subsequently decay into particles of masses m_2 and m_3, yielding our three-body final state. This is shown in Figure 5.

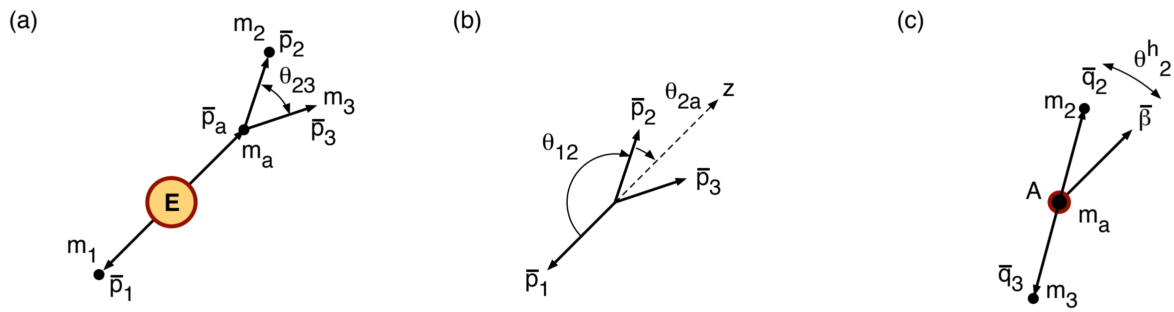

Figure 5: (a) Shows a system at rest of total energy E that decays into a particle of mass m_a and one of mass m_1. The particle of mass m_a subsequently decays into a pair of daughter particles of masses m_2 and m_3. (b) Shows the momentum of the three final-state particles in the initial frame. We choose the z-axis along the direction of particle A, and particle 2 makes and angle of θ_{2a} from z. Similarly, the angle from 1 to 2, θ_{12} is just $\pi - \theta_{2a}$. (c) Shows the particle of mass m_a, in its rest frame, decaying into m_2 and m_3. The particle of mass m_2 makes an angle of θ_2^h with respect to the direction of the particle of mass m_a (the direction of the Lorentz boost to move from the initial frame to the rest frame of the decaying particle. This frame is known as the helicity frame.

In the language of invariant masses, $m_a = m_{23}$, so we have a particle of mass m_{23} recoiling against particle of mass m_1. In our Dalitz plot (Figure 4), this would appear as a vertical band of fixed m_{23}^2. We would now like to examine the the distribution in m_{12}^2 as we move along the vertical band in fixed m_{23}^2 from the minimum to to the maximum values of m_{12}^2.

In the center-of-mass frame, the total energy is E, and we define a coordinate system whose z-axis is along $\vec{p}_a$ and whose y-axis is normal to the decay plane of the three particles. We then want to boost along $\hat{z}$ into the rest frame of m_{23}, and look at the angle between particle 2 and the boost direction, $\hat{\beta}$. We will call this angle θ_2^h, as shown in Figure 5(c), and we are interested in the distribution in the cosine of this angle, $\cos\theta_2^h$.

In this frame, we can write the momentum of particle 2 as

$$\vec{p}_2 = \begin{pmatrix} p_2 \sin\theta_{2a} \\ 0 \\ p_2 \cos\theta_{2a} \end{pmatrix}$$

where θ_{2a} is the angle between $\vec{p}_2$ and $\vec{p}_a$, as shown in Figure 5(b). The Lorentz boost factor β into the rest frame of m_{23} is given as

$$\vec{\beta} = \frac{\vec{p}_a}{E_a}.$$

Noting that $E_a = E - E_1$, and that $\vec{p}_1 = -\vec{p}_a$, we can write

$$\begin{aligned} \vec{\beta} &= -\frac{p_1}{E - E_1}\,\hat{z} \\ \gamma &= \frac{E - E_1}{m_{23}} \\ \beta\gamma &= \frac{p_1}{m_{23}} \end{aligned}$$

This yields the energy, E_2^h, and momentum, $\vec{p}_2^{\,h}$, of particle 2 in the helicity frame as

$$\begin{pmatrix} E_2^h \\ p_{2x}^h \\ p_{2z}^h \end{pmatrix} = \begin{pmatrix} \gamma & 0 & -\beta\gamma \\ 0 & 1 & 0 \\ -\beta\gamma & 0 & \gamma \end{pmatrix} \begin{pmatrix} E_2 \\ p_2 \sin\theta_{12} \\ p_2 \cos\theta_{12} \end{pmatrix} .$$

$$\begin{aligned} E_2^h &= \gamma E_2 - \beta\gamma p_2 \cos\theta_{12} \\ p_{2x}^h &= p_2 \sin\theta_{12} \qquad (19) \\ p_{2z}^h &= \gamma p_2 \cos\theta_{12} - \beta\gamma E_2 \, . \qquad (20) \end{aligned}$$

For a given m_{23}, the magnitude of $\vec{p}_2^{\,h}$ and the energy, E_2^h, can be obtained from two-body decay kinematics. We have a particle of mass m_{23} decaying to a pair of daughters of masses m_2 and m_3, so we have

$$\begin{aligned} p_2^h &= \sqrt{\frac{(m_{23}^2 + m_3^2 - m_2^2)^2}{4m_{23}^2} - m_3^2} \\ E_2^h &= \sqrt{\frac{(m_{23}^2 - m_3^2 + m_2^2)^2}{4m_{23}^2}} \end{aligned}$$

We know that $\cos\theta_2$, in the helicity frame, can be found from equation 19.

$$\cos\theta_2^h = \frac{\gamma p_2 \cos\theta_{2a} \; - \; \beta\gamma E_2}{p_2^h}$$

This can be simplified by inserting γ and $\beta\gamma$ from above into the equation, and then using equation 12 to replace write

$$p_2 \cos\theta_{12} = -\frac{m_{12}^2 - m_1^2 - m_2^2 - 2E_1E_2}{2p_1} \, .$$

Since $\theta_{12} = \pi - \theta_{2a}$, we have

$$p_2 \cos\theta_{2a} = \frac{m_{12}^2 - m_1^2 - m_2^2 - 2E_1E_2}{2p_1} \, .$$

After these substitutions, we obtain

$$\cos\theta_2^h = \frac{1}{p_2^h}\left[\left(\frac{E - E_1}{m_{23}}\right)\left(\frac{m_{12}^2 - m_1^2 - m_2^2 - 2E_1E_2}{2p_1}\right) - \left(\frac{p_1}{m_{23}}\right) E_2\right] .$$

Expanding this, and using $E_1^2 - p_1^2 = m_1^2$, we get

$$\cos\theta_2^h = \frac{E\left(m_{12}^2 - m_1^2 - m_2^2\right) - 2EE_1E_2 - E_1\left(m_{12}^2 - m_1^2 - m_2^2\right) + 2E_2m_1^2}{2p_1\, m_{23}\, p_2^h}$$

and then multiplying by $2E/2E$, and using equations 14 and 15 and

$$\begin{aligned} \cos\theta_2^h = & \frac{1}{4\,E\,p_1\,m_{23}\,p_2^h}\left[2E^2\left(m_{12}^2 - m_1^2 - m_2^2\right) - \left(E^2 + m_1^2 - m_{23}^2\right)\left(E^2 + m_2^2 - m_{13}^2\right)\right. \\ & - \left.\left(E^2 + m_1^2 - m_{23}^2\right)\left(m_{12}^2 - m_1^2 - m_2^2\right) + 2\left(E^2 + m_2^2 - m_{13}^2\right) m_1^2\right] \end{aligned}$$

From equation 13, we have

$$E^2 + m_2^h - m_{13}^2 = m_{12}^2 + m_{23}^2 - m_1^2 - m_3^2 \, .$$

$$\cos\theta_2^h = \frac{1}{4\,E\,p_1\,m_{23}\,p_2^h}\left[2E^2\left(m_{12}^2-m_1^2-m_2^2\right)-\left(E^2+m_1^2-m_{23}^2\right)\left(m_{12}^2+m_{23}^2-m_1^2-m_3^2\right)\right.$$
$$\left.-\left(E^2+m_1^2-m_{23}^2\right)\left(m_{12}^2-m_1^2-m_2^2\right)+2\left(m_{12}^2+m_{23}^2-m_1^2-m_3^2\right)m_1^2\right]$$

$$\cos\theta_2^h = \frac{\left[-2m_{23}^2\right]m_{12}^2+\left[m_{23}^2(m_2^2+m_3^2-m_{23}^2)+m_1^2(m_2^2-m_3^2+m_{23}^2)-E^2(m_2^2-m_3^2-m_{23}^2)\right]}{4Ep_2^h p_1 m_{23}}$$

Which for a fixed value of m_{23}^2, p_1, and p_2^h implies that $\cos\theta_2^h$ is a linear function of m_{12}^2. We can now substitute in for the values of p_1 and p_2^h which leads to the following.

$$\cos\theta_2^h = \frac{\left[-2m_{23}^2\right]m_{12}^2+\left[m_{23}^2(m_2^2+m_3^2-m_{23}^2)+m_1^2(m_2^2-m_3^2+m_{23}^2)-E^2(m_2^2-m_3^2-m_{23}^2)\right]}{\sqrt{\left[E^4+(m_1^2-m_{23}^2)^2-2E^2(m_1^2+m_{23}^2)\right]\left[m_2^4+(m_{23}^2-m_3^2)^2-2m_2^2(m_{23}^2+m_3^2)\right]}}$$

For a vertical band in m_{23}^2 in Figure 4, $\cos\theta_2^h$ would be $+1$ and the bottom of the band and -1 at the top of the band.

4 The Helicity Formalism for Decays

4.1 The decay $A \to B\,C$

First we consider a particle A of spin J and spin projection M along some arbitrary axis. In its rest frame, A decays into two daughter particles B and C with spins s_1, s_2. The two particles have equal and opposite momentum, p, and move along a direction $\hat{n}(\theta,\phi)$ with respect to the spin-quantization axis (z) of particle A. The final state can then be described by $(2s_1+1)\cdot(2s_2+1)$ helicity states $|p\lambda_1\lambda_2\rangle$. The helicities of the daughter particles, λ_i are defined as the projection of their total angular momentum , $\vec{J}=\vec{l}=\vec{s}$, along the direction of flight [4, 5] as measured in the rest frame of A.

$$\lambda = \vec{J}\cdot\frac{\vec{p}}{|\vec{p}|} = l\cdot\frac{\vec{p}}{|\vec{p}|}+m_s = m_s \tag{21}$$

Taking note of the fact that p is fixed, we can write that the amplitude for the decay, $\mathcal{A}$ can be expressed as;

$$\mathcal{A} = <\theta,\phi,\lambda_1,\lambda_2 \mid U \mid JM> \quad . \tag{22}$$

The probability of a daughter emerging with polar angle (θ,ϕ) is given as $\mid\mathcal{A}\mid^2$, which means that if we can calculate $\mathcal{A}$, then we can determine the angular distribution.

The rotation operator, $R(\alpha,\beta,\gamma)$ is normally expressed in terms of the Euler angles, where the total rotation can be expressed as the product of three individual rotations. The first is by and angle α around the z axis, the second is by an angle β around the new y axis, and the third is by an angle γ around the final z axis:

$$R(\alpha,\beta,\gamma) = R_z(\gamma)R_y(\beta)R_z(\alpha) = e^{-i\gamma J_z}e^{-i\beta J_y}e^{-i\alpha J_z} \tag{23}$$

where we use the fact that a rotation about some axis $\hat{n}$ is generated by the angular momentum operator $\vec{J}\cdot\hat{n}$. The angular momentum eigenstates $\mid jm>$ transform irreducibly under rotations because R and J^2 commute. A representation is labeled by the total angular momentum, j, and the action of $R(\alpha,\beta,\gamma)$ on a basis state, $\mid jm>$ is:

$$R(\alpha,\beta,\gamma)\mid jm> = \sum_{m'=-j}^{j} D^j_{mm'}(\alpha,\beta,\gamma)\mid jm'> \tag{24}$$

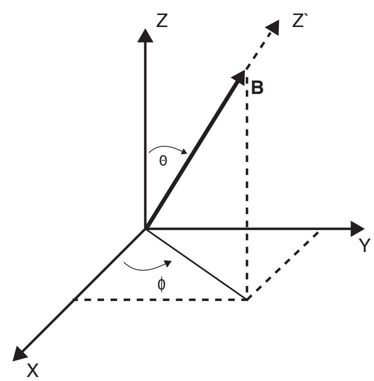

Figure 6: Coordinate system for the decay $A \to BC$

We can now multiply this by the bra-state $< jm'' \mid$, and obtain:

$$< jm'' \mid R(\alpha,\beta,\gamma) \mid jm > \;=\; \sum_{m'=-j}^{j} D^{j}_{mm'}(\alpha,\beta,\gamma) < jm'' \mid jm' > \tag{25}$$

$$= D^{j}_{m''m'}(\alpha,\beta,\gamma) \;. \tag{26}$$

If we define :

$$d^{j}_{m'm}(\beta) =< jm' \mid e^{-i\beta J_y} \mid jm >,$$

then

$$D^{j}_{mm'}(\alpha\beta\gamma) = e^{-i\alpha m'} d^{j}_{m'm}(\beta) e^{-i\gamma m}.$$

Using this information, we now want to evaluate the amplitude as given in equation 22. While the two body helicity wave function, $\mid \theta,\phi,\lambda_1,\lambda_2 >$ is not itself an eigenfunction of J^2, we can express it in terms of a sum over eigenstates of J^2. Namely,

$$\mid \theta,\phi,\lambda_1,\lambda_2 > \;=\; \sum_{J,M} \sqrt{\frac{2J+1}{4\pi}} D^{J}_{M\lambda}(\phi,\theta,-\phi) \mid J, M, \lambda_1\lambda_2 > \tag{27}$$

where $\lambda = \lambda_1 - \lambda_2$. Equation 22 can now be written as:

$$\begin{aligned} f_{\lambda_1\lambda_2 M}(\theta,\phi) &= < \theta,\phi,\lambda_1,\lambda_2 \mid U \mid JM > \\ &= \sum_{J'M'} < \theta,\phi,\lambda_1,\lambda_2 \mid J_f, M_f, \lambda_1,\lambda_2 >< J_f, M_f, \lambda_1,\lambda_2 \mid U \mid JM > \\ &= \sqrt{\frac{2J+1}{4\pi}} D^{J*}_{M\lambda}(\phi,\theta,-\phi) T_{\lambda_1\lambda_2} \,. \end{aligned}$$

It should be pointed out that there is often a degree of choice for the third rotation angle. Two common choices are $-\phi$, and 0. Under the second choice, one often sees the D-function written as $D^{J*}_{M\lambda}(\theta,\phi)$. It is worth mentioning that for reactions in which the final state particles have non-zero spin, the choice of the third angle is no longer arbitrary, and care needs to be taken. Under the former choice of $-\phi$ for the third angle, it is easy to rewrite f as:

$$f_{\lambda_1\lambda_2 M}(\theta,\phi) \quad = \quad \sqrt{\frac{2J+1}{4\pi}} D^J_{\lambda,M}(-\phi,-\theta,\phi) T_{\lambda_1\lambda_2}\,.$$

The interaction is rotation invariant. The transition amplitude is a matrix with $(2s_1{+}1)(2s_2{+}1)$ rows and $(2J{+}1)$ columns. $D^{J*}_{M\lambda}(\phi,\theta,-\phi)$ describes the geometry (the rotation of the system Σ_3 where the helicity states are defined back into the CMS system of the resonance) $T_{\lambda_1\lambda_2}$ describes the dependences from the spins and the orbital angular momenta of the different particles in the decay process. The general form of $T_{\lambda_1\lambda_2}$ is given by

$$T_{\lambda_1\lambda_2} = \sum_{ls} \alpha_{ls} \langle J\lambda | ls0\lambda\rangle \langle s\lambda | s_1 s_2 \lambda_1, -\lambda_2\rangle \tag{28}$$

where α_{ls} are unknown fit parameters. They define the decay configuration concerning spin and orbital angular momentum into one specific decay channel. The brackets are Clebsch-Gordan coefficients which describe the couplings $\vec{J} = \vec{l} + \vec{s}$ und $\vec{s} = \vec{s}_1 + \vec{s}_2$. It is summed over all l and s possible by angular momentum, parity and C-parity conservation. Finally the angular distribution is obtained by calculating:

$$w_D(\theta,\phi) = Tr(\rho_f) = Tr(f\rho_i f^+) \tag{29}$$

ρ_f is the final state density matrix of the dimension $(2s_1{+}1)(2s_2{+}1)$ and ρ_i is the initial density matrix of dimension (2J+1).

- Multiple decay chains

One of the advantages of the helicity formalism is that it can easily be extended to successive decays. This is in principle possible for an arbitrary number of further two-body decays; of course the calculation of the decay angular distribution gets more and more complex. Assume that not only A decays into B and C but also B and C decay further into B_1B_2 and C_1C_2. The total helicity amplitude for a reaction $A \to BC$, $B \to B_1B_2$, $C \to C_1C_2$, has the following form:

$$\begin{aligned} f_{tot} &= [f(B) \otimes f(C)]\, f(A) \\ &= \sum_{\lambda(B)\lambda(C)} \left[f_{\lambda(B1)\lambda(B2),\lambda(B)} \otimes f_{\lambda(C1)\lambda(C2),\lambda(C)}\right] f_{\lambda(B)\lambda(C),\lambda(A)} \end{aligned}$$

$\otimes$ represents the tensor product of two matrices. To calculate the angular distribution of a complicated decay chain the transition amplitude for the different single decays $f_{\lambda(x1)\lambda(x2),\lambda(x)}$ of $X \to X_1X_2$ are calculated first. These are then combined. Fig. 7 shows in which cases this combination is done by a scalar or by a tensor product.

All decays which are placed in the figure one besides another have to be combined by a tensor product (all $(J,m_J)_A$ with all $(J,m_J)_B$); all decays which happen in a line are combined by a scalar product. Here the $(J,m_J)_{A1}$, depends directly on $(J,m_J)_A$. For Fig. 7 one gets:

$$f_{tot} = [\ (\ f(A2)f(A1)f(A)\ ) \otimes (\ [\ f(B2) \otimes f(B1)\ ]\ f(B)\ )\ ]\, f(\bar{p}N) \tag{30}$$

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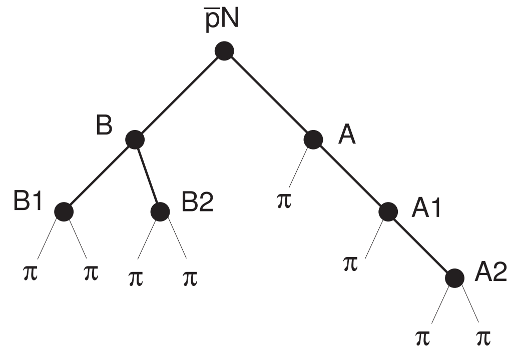

Figure 7: Decay chain

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