AmpTools Implementation
of Polarized Production
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For the reaction $\vec{\gamma}p \to \eta\pi p$ we seek to formulate the intensity in the reflectivity basis in such a way that it is easily incorporated into the AmpTools package for amplitude analysis. The intensity describes the dependence of the cross section on various kinematic variables:

$$I(\Omega, \Phi) = \frac{d\sigma}{dt\ dm_{\eta\pi}\ d\Omega\ d\Phi}, \tag{1}$$

where Φ is the angle of the beam photon polarization with respect to the production plane and Ω is the direction of η in the helicity frame. We follow the development in V. Mathieu *et al.* (arXiv:1906.04841), where Eq. 3 is

$$I(\Omega, \Phi) = I^0(\Omega) - P_\gamma I^1(\Omega)\cos 2\Phi - P_\gamma I^2(\Omega)\sin 2\Phi. \tag{2}$$

Here P_γ is the fraction of linear polarization of the beam, and $0 \leq P_\gamma \leq 1$. We suppress the helicity indices of the target and recoil nucleon in what follows, and then re-introduce them later when formulating the final expression for the intensity.

Again, following Mathieu *et al.*, we have

$$I^0(\Omega) = \frac{\kappa}{2}\sum_\lambda A_\lambda(\Omega)A^*_\lambda(\Omega), \tag{3}$$

$$I^1(\Omega) = \frac{\kappa}{2}\sum_\lambda A_{-\lambda}(\Omega)A^*_\lambda(\Omega), \tag{4}$$

$$I^2(\Omega) = i\frac{\kappa}{2}\sum_\lambda \lambda A_{-\lambda}(\Omega)A^*_\lambda(\Omega). \tag{5}$$

In the expressions above λ indexes the photon beam helicity, which takes on the values of $+, -$. We can then rewrite the intensity as

$$\begin{aligned} I(\Omega, \Phi) &= \frac{\kappa}{2}\big[A_-(\Omega)A^*_-(\Omega) + A_+A^*_+(\Omega) - P_\gamma\cos 2\Phi\left(A_-(\Omega)A^*_+(\Omega) + A_+(\Omega)A^*_-(\Omega)\right) - \\ &\quad iP_\gamma \sin 2\Phi\left(A_-(\Omega)A^*_+(\Omega) - A_+(\Omega)A^*_-(\Omega)\right)\big] \\ &= \frac{\kappa}{2}\left(A_-(\Omega)A^*_-(\Omega) + A_+A^*_+(\Omega) - P_\gamma e^{i2\Phi}A_-(\Omega)A^*_+(\Omega) - P_\gamma e^{-i2\Phi}A_+(\Omega)A^*_-(\Omega)\right). \end{aligned} \tag{6}$$

Now define the amplitudes $\tilde{A}_+(\Omega)$ and $\tilde{A}_-(\Omega)$:

$$\tilde{A}_\pm(\Omega, \Phi) \equiv e^{\mp i\Phi}A_\pm(\Omega). \tag{7}$$

In terms of these amplitudes the intensity becomes

$$\begin{aligned} I(\Omega, \Phi) &= \frac{\kappa}{2}\big(\tilde{A}_-(\Omega, \Phi)\tilde{A}^*_-(\Omega, \Phi) + \tilde{A}_+(\Omega, \Phi)\tilde{A}^*_+(\Omega, \Phi) - \\ &\quad P_\gamma\tilde{A}_-(\Omega, \Phi)\tilde{A}^*_+(\Omega, \Phi) - P_\gamma\tilde{A}_+(\Omega, \Phi)\tilde{A}^*_-(\Omega, \Phi)\big) \\ &= \frac{\kappa}{4}\left[(1 - P_\gamma)\left|\tilde{A}_+(\Omega, \Phi) + \tilde{A}_-(\Omega, \Phi)\right|^2 + (1 + P_\gamma)\left|\tilde{A}_+(\Omega, \Phi) - \tilde{A}_-(\Omega, \Phi)\right|^2\right]. \end{aligned} \tag{8}$$

We now would like to expand the amplitudes in partial waves in the beam photon helicity basis and then transform to the reflectivity basis. Again, following the notation of Mathieu *et al.* Eq. 5 (and suppressing the nucleon helicity indices):

$$A_\lambda(\Omega) = \sum_{\ell,m} T^\ell_{\lambda m} Y^m_\ell(\Omega). \tag{9}$$

Using Eqs. D2a and D2b from the reference, which express the helicity amplitudes $T^{\ell}_{\lambda,m}$ in terms of reflectivity amplitudes ${}^{(\epsilon)}T^{\ell}_{m}$, we can write expressions for the partial wave expansions of $\tilde{A}_{\pm}(\Omega,\Phi)$.

$$\begin{aligned}
\tilde{A}_{+}(\Omega,\Phi) &= e^{-i\Phi}\sum_{\ell,m}\left[{}^{(-)}T^{\ell}_{m} +^{(+)} T^{\ell}_{m}\right] Y^{m}_{\ell}(\Omega) \\
&= \sum_{\ell,m}\left[{}^{(-)}T^{\ell}_{m} +^{(+)} T^{\ell}_{m}\right] Z^{m}_{\ell}(\Omega,\Phi) \qquad (10)\\
\tilde{A}_{-}(\Omega,\Phi) &= e^{i\Phi}\sum_{\ell,m}(-1)^{m}\left[{}^{(-)}T^{\ell}_{-m} -^{(+)} T^{\ell}_{-m}\right] Y^{m}_{\ell}(\Omega) \\
&= e^{i\Phi}\sum_{\ell,m}\left[{}^{(-)}T^{\ell}_{m} -^{(+)} T^{\ell}_{m}\right] (-1)^{m} Y^{-m}_{\ell}(\Omega) \\
&= e^{i\Phi}\sum_{\ell,m}\left[{}^{(-)}T^{\ell}_{m} -^{(+)} T^{\ell}_{m}\right] Y^{m*}_{\ell}(\Omega) \\
&= \sum_{\ell,m}\left[{}^{(-)}T^{\ell}_{m} -^{(+)} T^{\ell}_{m}\right] Z^{m*}_{\ell}(\Omega,\Phi), \qquad (11)
\end{aligned}$$

where in the last line of each expression we have defined phase-rotated spherical harmonics:

$$Z^{m}_{\ell}(\Omega,\Phi) \equiv Y^{m}_{\ell}(\Omega)e^{-i\Phi}. \qquad (12)$$

Now we can write expression for the sums and differences that appear in the intensity

$$\tilde{A}_{+}(\Omega,\Phi) + \tilde{A}_{-}(\Omega,\Phi) = 2\sum_{\ell,m}\left({}^{(-)}T^{\ell}_{m}\mathrm{Re}[Z^{m}_{\ell}(\Omega,\Phi)] + i\ {}^{(+)}T^{\ell}_{m}\mathrm{Im}[Z^{m}_{\ell}(\Omega,\Phi)]\right) \qquad (13)$$

$$\tilde{A}_{+}(\Omega,\Phi) - \tilde{A}_{-}(\Omega,\Phi) = 2\sum_{\ell,m}\left({}^{(+)}T^{\ell}_{m}\mathrm{Re}[Z^{m}_{\ell}(\Omega,\Phi)] + i\ {}^{(-)}T^{\ell}_{m}\mathrm{Im}[Z^{m}_{\ell}(\Omega,\Phi)]\right) \qquad (14)$$

We use these expressions to write the intensity as

$$\begin{aligned}
I(\Omega,\Phi) = \kappa(1-P_{\gamma})&\left|\sum_{\ell,m}\left({}^{(-)}T^{\ell}_{m}\mathrm{Re}[Z^{m}_{\ell}(\Omega,\Phi)] + i\ {}^{(+)}T^{\ell}_{m}\mathrm{Im}[Z^{m}_{\ell}(\Omega,\Phi)]\right)\right|^{2} + \\
\kappa(1+P_{\gamma})&\left|\sum_{\ell,m}\left({}^{(+)}T^{\ell}_{m}\mathrm{Re}[Z^{m}_{\ell}(\Omega,\Phi)] + i\ {}^{(-)}T^{\ell}_{m}\mathrm{Im}[Z^{m}_{\ell}(\Omega,\Phi)]\right)\right|^{2}. \qquad (15)
\end{aligned}$$

Up to this point, we have ignored the sum over target and recoil nucleon helicites, λ_1 and λ_2, respectively. In order to simplify the expression further, we need to reintroduce the sum over helicity and note Eq. D5 of the reference, which is derived from parity conservation:

$$\sum_{\lambda_1,\lambda_2} {}^{(\epsilon)}T^{\ell}_{m;\lambda_1\lambda_2}\ {}^{(\epsilon')}T^{\ell'*}_{m';\lambda_1\lambda_2} = (1+\epsilon\epsilon')\sum_{k}[\ell]^{(\epsilon)}_{m;k}[\ell']^{(\epsilon')*}_{m';k}. \qquad (16)$$

Here $[\ell]^{(\epsilon)}_{m;k}$ are the partial wave amplitudes in the reflectivity basis for spin flip ($k = 1$) and spin non-flip ($k = 0$) at the nucleon vertex. When summing over helicities, the products of amplitudes of opposite reflectivity vanish. Considering, for example, one of the coherent sums in the intensity, we can use this cancellation to rewrite it as two coherent sums over partial waves of definite reflectivity for nucleon spin flip and nucleon spin non-flip amplitudes (four coherent sums total):

$$\sum_{\lambda_1,\lambda_2}\left|\sum_{\ell,m}\left({}^{(-)}T^{\ell}_{m;\lambda_1\lambda_2}\mathrm{Re}[Z^m_\ell] + i\ {}^{(+)}T^{\ell}_{m;\lambda_1\lambda_2}\mathrm{Im}[Z^m_\ell]\right)\right|^2 =$$

$$\sum_{\lambda_1\lambda_2}\sum_{\ell,m}\sum_{\ell',m'}\left({}^{(-)}T^{\ell}_{m;\lambda_1\lambda_2}\mathrm{Re}[Z^m_\ell] + i\ {}^{(+)}T^{\ell}_{m;\lambda_1\lambda_2}\mathrm{Im}[Z^m_\ell]\right)\times\left({}^{(-)}T^{\ell'*}_{m';\lambda_1\lambda_2}\mathrm{Re}[Z^{m'}_{\ell'}] - i\ {}^{(+)}T^{\ell'*}_{m';\lambda_1\lambda_2}\mathrm{Im}[Z^{m'}_{\ell'}]\right)$$

$$= 2\sum_k\sum_{\ell,m}\sum_{\ell',m'}\left([\ell]^{(+)}_{m;k}[\ell']^{(+)*}_{m';k}\mathrm{Im}[Z^m_\ell]\mathrm{Im}[Z^{m'}_{\ell'}] + [\ell]^{(-)}_{m;k}[\ell']^{(-)*}_{m';k}\mathrm{Re}[Z^m_\ell]\mathrm{Re}[Z^{m'}_{\ell'}]\right)$$

$$= 2\sum_k\left(\left|\sum_{\ell,m}[\ell]^{(+)}_{m;k}\mathrm{Im}[Z^m_\ell]\right|^2 + \left|\sum_{\ell,m}[\ell]^{(-)}_{m;k}\mathrm{Re}[Z^m_\ell]\right|^2\right). \tag{17}$$

Finally we arrive at a formulation of the intensity that involves four coherent sums for each configuration of nucleon spin:

$$I(\Omega,\Phi) = 2\kappa\sum_k\Bigg\{(1-P_\gamma)\left|\sum_{\ell,m}[\ell]^{(-)}_{m;k}\mathrm{Re}[Z^m_\ell(\Omega,\Phi)]\right|^2 + (1-P_\gamma)\left|\sum_{\ell,m}[\ell]^{(+)}_{m;k}\mathrm{Im}[Z^m_\ell(\Omega,\Phi)]\right|^2 + (1+P_\gamma)\left|\sum_{\ell,m}[\ell]^{(+)}_{m;k}\mathrm{Re}[Z^m_\ell(\Omega,\Phi)]\right|^2 + (1+P_\gamma)\left|\sum_{\ell,m}[\ell]^{(-)}_{m;k}\mathrm{Im}[Z^m_\ell(\Omega,\Phi)]\right|^2\Bigg\}. \tag{18}$$

These expressions can be easily implemented into AmpTools by absorbing a factor of $\sqrt{1 \pm P_\gamma}$ into the definition of the amplitude. One would then fit for the $[\ell]^{(\pm)}_{m;k}$ coefficients in the four separate coherent sums, where repeated $[\ell]^{(\pm)}_{m;k}$ are constrained to be the same using the `constrain` keyword in the AmpTools configuration file. The overall factor 2κ is irrelevant as the intensity will be renormalized in the fitting procedure. It would be useful to define an amplitude for the function Z^m_ℓ with appropriate flags such that the `calcAmplitude` function returns, as a complex double data type, the real or imaginary part of Z^m_ℓ (which is a purely real number).

Since there are no kinematic variables to distinguish between the spin flip ($k = 1$) and spin non-flip ($k = 0$) amplitudes, it is anticipated that these two contributions to the intensity will result in an poorly constrained fit. The contributions from two k values are not purely ambiguous since the coherent sums for each k may contain a different set of interfering amplitudes depending that depends on k. Nevertheless, fit instability seems highly likely for the general case where both $k = 0$ and $k = 1$ is included for all amplitudes.

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