The Photon Spin Density Matrix

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Contents

1 Density Matrix

Given a set of n quantum states $|\psi_n\rangle$ with statistical weights W_n, the general definition of the density matrix is:

$$\rho = \sum_n W_n |\psi_n\rangle\langle\psi_n|. \tag{1}$$

and since we can write any state $|\psi_n\rangle$ in any orthogonal basis $|\phi_i\rangle$ as:

$$|\psi_n\rangle = \sum_i a_i^n |\phi_i\rangle. \tag{2}$$

$$\langle\psi_n| = \sum_{i'} a_{i'}^{n*} \langle\phi_{i'}|. \tag{3}$$

Then,

$$\rho = \sum_n \sum_{i,i'} W_n a_i^n a_{i'}^{n*} |\phi_i\rangle\langle\phi_{i'}|. \tag{4}$$

with.

$$\delta_{i,i'} = \langle\phi_{i'}|\phi_i\rangle. \tag{5}$$

and

$$1 = \sum_i |\phi_i\rangle\langle\phi_i|. \tag{6}$$

therefore the element i, j of the density matrix is

$$\rho_{ij} = \langle\phi_i|\rho|\phi_j\rangle = \sum_n W_n a_i^n a_j^{n*}. \tag{7}$$

The density matrix is hermitian since

$$\langle \phi_i|\rho|\phi_j\rangle = \langle \phi_j|\rho|\phi_i\rangle^*. \tag{8}$$

The probability of finding the system in the state $|\phi_m\rangle$ is given by the diagonal element

$$\rho_{mm} = \sum_n W_n |a_m^n|^2. \tag{9}$$

Therefore the diagonal elements are positive and real.
The probability, $W(\psi)$, of finding the system in the state $|\psi\rangle$ after a measure is done is given by

$$W(\psi) = \langle \psi|\rho|\psi\rangle = \sum_n W_n |\langle \psi_n|\psi\rangle|^2. \tag{10}$$

We can calculate the trace of the density matrix by

$$tr\rho = \sum_i \rho_{ii} = \sum_n W_n \sum_i |a_i^n|^2 = 1. \tag{11}$$

since $\sum_n W_n = 1$ and $\sum_i |a_i^n|^2 = 1$.
The expectation value of any operator, $\langle \mathbb{O}\rangle$, can be written as

$$\langle \mathbb{O}\rangle = \sum_n \sum_{i,i'} W_n a_i^n a_{i'}^{n*} \langle \phi_{i'}|\mathbb{O}|\phi_i\rangle = \sum_{i,i'} \langle \phi_{i'}|\rho|\phi_i\rangle \langle \phi_{i'}|\mathbb{O}|\phi_i\rangle. \tag{12}$$

Therefore

$$\langle \mathbb{O}\rangle = tr(\rho\mathbb{O}). \tag{13}$$

We will often encounter density matrices of 2x2 dimensions (for example spin 1/2 particles and the real photon). In those cases is very useful to consider the Pauli matrices as a particular basis for the density matrix in that space. The Pauli matrices are $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It can be shown that their algebra is defined by:

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + i\sum_k \epsilon_{ijk}\sigma_k \tag{14}$$

where δ_{ij} is the Kronecker symbol and ϵ_{ijk} is the Levi-Civita symbol. It is easy to show from that relations that $tr(\sigma_i) = 0$ and $tr(\sigma_i\sigma_j) = 2\delta_{ij}$. Any 2x2 hermitian matrix could be written in the Pauli matrix basis plus a unit matrix (the unit matrix needed to obtain the four independent parameters needed to determine the density matrix). In particular, the density matrix can be written as

$$\rho = b_0\mathbf{1} + \sum_{k=1,2,3} b_k\sigma_k \tag{15}$$

where $b_k, k = 0,1,2,3$ are the parameters of the expansion. Taken the trace we obtain

$$tr(\rho) = 2b_0 = 1 \tag{16}$$

Therefore, $b_0 = \frac{1}{2}$. To obtain the other parameters let's calculate:

$$tr(\rho\sigma_i) = \langle\sigma_i\rangle = tr(\frac{1}{2}\sigma_i + \sum_{k=1,2,3} b_k\sigma_k\sigma_i) = \frac{1}{2}tr\sigma_i + \sum_{k=1,2,3} b_k tr(\sigma_k\sigma_I) \tag{17}$$

Therefore

$$tr(\rho\sigma_i) = \langle\sigma_i\rangle = 2b_i \tag{18}$$

we then write

$$\rho = \frac{1}{2}\left(\mathbf{1} + \textstyle\sum_{k=1,2,3}\langle\sigma_k\rangle\sigma_k\right) \tag{19}$$

For the three component vector, $b_i = \langle\sigma_i\rangle$, we will use the notation $\overrightarrow{\mathscr{P}}$. It contains all polarization information (NOTE: some authors refer to this vector as "polarization vector", we will keep that name to the more classical formulation that will be later defined) then

$$\mathscr{P}_i = \langle\sigma_i\rangle \tag{20}$$

And

$$\rho = \frac{1}{2}\left(\mathbf{1} + \textstyle\sum_{k=1,2,3}\mathscr{P}_k\sigma_k\right) = \frac{1}{2}\left(\mathbf{1} + \overrightarrow{\mathscr{P}}\cdot\boldsymbol{\sigma}\right) \tag{21}$$

And in matrix form

$$\rho = \frac{1}{2}[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \mathscr{P}_1\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \mathscr{P}_2\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \mathscr{P}_3\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}] \tag{22}$$

$$\rho = \frac{1}{2}\begin{pmatrix}1+\mathscr{P}_3 & \mathscr{P}_1 - i\mathscr{P}_2\\ \mathscr{P}_1 + i\mathscr{P}_2 & 1-\mathscr{P}_3\end{pmatrix} \tag{23}$$

2 Photon Spin Density Matrix

Consider the reaction

$$\vec{\gamma}N \to XN' \tag{24}$$

where $\vec{\gamma}$ is a linearly polarized photon beam, N, N' are nucleons and X is a mesonic resonance. The form of the scattering amplitude $\mathscr{M}$, represented by the transition operator, $\widehat{T}$, given by

$$\mathscr{M} = \langle out|T|in\rangle \tag{25}$$

and then the number of events will be proportional to:

$$I(\tau) = \sum_{ext.-spins}|\mathscr{M}|^2 = \sum_{ext.-spins}\langle out|T\rho_{in}T^\dagger|out\rangle = \sum_{ext.-spins}\sum_{i,j}\langle out|T\ \rho_{i,j}T^\dagger|out\rangle. \tag{26}$$

The operator $|in\rangle\langle in|$ is defined as the initial spin density matrix operator, ρ_{in}, where the indices run over initial spins

$$\rho_{in} = |in\rangle\langle in|. \tag{27}$$

ρ_{in} includes the beam spin information and it is defined as *the spin density matrix of the incoming photon* ρ_γ.
Since the photon wave is transverse (Lorentz condition), any polarization state will be in a plane perpendicular to the direction of the photon momentum. Therefore, the polarization of real photons could be written in a two vectors base or it will have two possible pure spin states. In a classical view, one could coincide with the electric field direction. Any polarization direction can be represented by the superposition of these two orthogonal (pure) states contained in the transverse plane. Let's take these two states to be $|P_1\rangle$, in the direction of a pure state (for example the direction of the electric vector of the prepared incoming photon). This vector is called *polarization vector*. $|P_2\rangle$ is taken orthogonal to the $|P_1\rangle$ state, as the basis.
We found that in the $|P_1\rangle, |P_2\rangle$ basis, the spin density matrix of a mixed polarization state (superposition of two pure polarization states), can be written in the formal notation, where W_1 and W_2 are the weights for each state [1]

$$\rho_\gamma = W_1|P_1\rangle\langle P_1| + W_2|P_2\rangle\langle P_2|. \tag{28}$$

Consider N beam photons. The meaning of (28) is that, when the beam polarization is measured, we will find N_1 photons polarized in the state with amplitude $\langle P_1|\rho_\gamma|P_1\rangle$, and N_2 in the state with amplitude $\langle P_2|\rho_\gamma|P_2\rangle$ such that $N_1 + N_2 = N$. Let's assume that $N_1 \geq N_2$, i.e. the index one corresponding to the $\overrightarrow{OX}$ axis is assigned to the maximum number of photons (assuming the opposite will produce a change of signs in the formulas with no physical consequences). We define the *partial polarization* (or *degree of polarization*), $\mathscr{P}$ (see below about adopting this notation), such that

$$\mathscr{P} = \frac{N_1 - N_2}{N}. \tag{29}$$

Notice that $0 \leq \mathscr{P} \leq 1$.
If all photons are found in the $\langle P_1|\rho_\gamma|P_1\rangle$ state, $N_1 = N$ then $\mathscr{P} = 1$ (full polarization), if all beam particles are found equally distributed between $\langle P_1|\rho_\gamma|P_1\rangle$ and $\langle P_2|\rho_\gamma|P_2\rangle$ (no polarization), $\mathscr{P} = 0$.
We can now calculate the weights (28) using the interpretation of probabilities as frequencies

$$W_1 = \frac{N_1}{N} \; and \; W_2 = \frac{N_2}{N} \tag{30}$$

and solving the system

$$\begin{aligned} N &= N_1 + N_2 \\ \mathscr{P}N &= N_1 - N_2 \end{aligned} \tag{31}$$

we obtain

$$W_1 = \frac{N_1}{N} = \frac{(1+\mathscr{P})}{2} \; and \; W_2 = \frac{N_2}{N} = \frac{(1-\mathscr{P})}{2} \tag{32}$$

therefore

$$\rho_\gamma = \frac{(1+\mathscr{P})}{2}|P_1\rangle\langle P_1| + \frac{(1-\mathscr{P})}{2}|P_2\rangle\langle P_2| \tag{33}$$

and in matrix form:

$$\rho_\gamma(\mathscr{P}) = {}^1\!/_2 \begin{pmatrix} 1+\mathscr{P} & 0 \\ 0 & 1-\mathscr{P} \end{pmatrix}. \tag{34}$$

Comparing this matrix with (27) we obtain:
$\overrightarrow{\mathscr{P}} = (0, 0, \mathscr{P})$
Therefore, the vector $\overrightarrow{\mathscr{P}}$ has the magnitude of the partial polarization. As demonstrated before, is generally written by the identity

$$\widehat{\rho_\gamma} = \frac{1}{2}I + \frac{1}{2}\overrightarrow{\mathscr{P}} \cdot \sigma \tag{35}$$

where I is the unit matrix (2×2), and the σ_i are the three Pauli matrices.
We will now calculate this matrix in three different bases: the canonical (using the Gottfried-Jackson (GJ) frame of reference), the helicity basis and finally in the reflectivity basis.
The Gottfried-Jackson frame (GJ) is a frame where the resonance (X) is at rest. z is in the direction of the beam and y is perpendicular to the production plane, such that $\overrightarrow{y} = \overrightarrow{p}_{Beam} \times \overrightarrow{p}_X$ and x is given to produce a right-handed system. The GJ angles, (θ_{GJ}, ϕ_{GJ}), are the standard polar coordinates in the GJ frame.. The photon density matrix could be written in a basis defined in that frame of reference (where there is a rotation about z - beam axis). To transform the matrix from the canonical with a frame where x is the pure polarization vector ($|P_1\rangle$) basis, as calculated in (34), to the one in a GJ frame ($|x\rangle$ and $|y\rangle$) we need to perform a rotation about the z axis (beam) of the form

$$\begin{aligned} |x\rangle &= \cos\Phi|P_1\rangle - \sin\Phi|P_2\rangle \\ |y\rangle &= \sin\Phi|P_1\rangle + \cos\Phi|P_2\rangle \end{aligned} \tag{36}$$

where Φ is the angle between (from) the production (hadronic) plane which is the x axis of the GJ frame and the polarization vector $|P_1\rangle$. In this frame the polarization vector has direction $(\cos\Phi, \sin\Phi, 0)$.
In this basis, the new matrix is

$$\rho_{x,y}(\mathscr{P}, \Phi) = \begin{pmatrix} \langle x|\rho|x\rangle & \langle x|\rho|y\rangle \\ \langle y|\rho|x\rangle & \langle y|\rho|y\rangle \end{pmatrix}. \tag{37}$$

We can calculate each element of the matrix using (36)

$$\langle x|\rho|x\rangle = (\cos\Phi, -\sin\Phi)\frac{1}{2}\begin{pmatrix} 1+\mathscr{P} & 0 \\ 0 & 1-\mathscr{P} \end{pmatrix}\begin{pmatrix} \cos\Phi \\ -\sin\Phi \end{pmatrix} \tag{38}$$

or

$$\langle x|\rho|x\rangle = \frac{(1+\mathscr{P})}{2}\cos^2\Phi + \frac{(1-\mathscr{P})}{2}\sin^2\Phi. \tag{39}$$

In the same way

$$\langle y|\rho|y\rangle = (\sin\Phi, \cos\Phi)\frac{1}{2}\begin{pmatrix} 1+\mathscr{P} & 0 \\ 0 & 1-\mathscr{P} \end{pmatrix}\begin{pmatrix} \sin\Phi \\ \cos\Phi \end{pmatrix} \tag{40}$$

or

$$\langle y|\rho|y\rangle = \frac{(1+\mathscr{P})}{2}\sin^2\Phi + \frac{(1-\mathscr{P})}{2}\cos^2\Phi. \tag{41}$$

The off-diagonal elements are (both elements are the same)

$$\langle x|\rho|y\rangle = (\cos\Phi, -\sin\Phi)\frac{1}{2}\begin{pmatrix} 1+\mathscr{P} & 0 \\ 0 & 1-\mathscr{P} \end{pmatrix}\begin{pmatrix} \sin\Phi \\ \cos\Phi \end{pmatrix} \tag{42}$$

therefore

$$\langle x|\rho|y\rangle = \frac{(1+\mathscr{P})}{2}\sin\Phi\cos\Phi - \frac{(1-\mathscr{P})}{2}\sin\Phi\cos\Phi. \tag{43}$$

In matrix form

$$\rho_{can}(\mathscr{P},\Phi) = \begin{pmatrix} \frac{(1+\mathscr{P})}{2}\cos^2\Phi + \frac{(1-\mathscr{P})}{2}\sin^2\Phi & \frac{(1+\mathscr{P})}{2}\sin\Phi\cos\Phi - \frac{(1-\mathscr{P})}{2}\cos\Phi\sin\Phi \\ \frac{(1+\mathscr{P})}{2}\cos\Phi\sin\Phi - \frac{(1-\mathscr{P})}{2}\sin\Phi\cos\Phi & \frac{(1+\mathscr{P})}{2}\sin^2\Phi + \frac{(1-\mathscr{P})}{2}\cos^2\Phi \end{pmatrix}. \tag{44}$$

After some algebra, the spin density matrix of the photon in the canonical basis (and GJ frame) becomes:

$$\rho_{can}(\mathscr{P},\Phi) = {}^1\!/_2\begin{pmatrix} 1+\mathscr{P}\cos 2\Phi & \mathscr{P}\sin 2\Phi \\ \mathscr{P}\sin 2\Phi & 1-\mathscr{P}\cos 2\Phi \end{pmatrix}. \tag{45}$$

Next, we transform the matrix to the helicity basis. We start by using the relations between the canonical basis and the helicity basis given by [?]

$$\begin{aligned} |\lambda=+1\rangle &= -\tfrac{1}{\sqrt{2}}(|x\rangle + i|y\rangle) \\ |\lambda=-1\rangle &= \tfrac{1}{\sqrt{2}}(|x\rangle - i|y\rangle) \end{aligned} \tag{46}$$

and using the relations in (36), we obtain

$$\begin{aligned} &|\lambda=+1\rangle = -\tfrac{1}{\sqrt{2}}\left[(\cos\Phi|P_1\rangle - \sin\Phi|P_2\rangle) + i\,(\sin\Phi|P_1\rangle + \cos\Phi|P_2\rangle)\right] \\ &= -\tfrac{1}{\sqrt{2}}\left[(\cos\Phi + i\sin\Phi)\,|P_1\rangle + (-\sin\Phi + i\cos\Phi)\,|P_2\rangle\right] \end{aligned} \tag{47}$$

therefore

$$|\lambda=+1\rangle = -\frac{1}{\sqrt{2}}\left(e^{i\Phi}|P_1\rangle + ie^{i\Phi}|P_2\rangle\right). \tag{48}$$

The other state of helicity is

$$|\lambda=-1\rangle=\tfrac{1}{\sqrt{2}}\left[(\cos\Phi|P_1\rangle-\sin\Phi|P_2\rangle)-i\left(\sin\Phi|P_1\rangle+\cos\Phi|P_2\rangle\right)\right]$$
$$=\tfrac{1}{\sqrt{2}}\left[(\cos\Phi-i\sin\Phi)\,|P_1\rangle-(\sin\Phi+i\cos\Phi)\,|P_2\rangle\right] \tag{49}$$

therefore

$$|\lambda=-1\rangle=\frac{1}{\sqrt{2}}\left(e^{-i\Phi}|P_1\rangle-ie^{-i\Phi}|P_2\rangle\right). \tag{50}$$

Then, we can calculate the elements of the new matrix:

$$\langle\lambda=+1|\rho|\lambda=+1\rangle=\left[(-\tfrac{1}{\sqrt{2}})(e^{-i\Phi},-ie^{-i\Phi})\tfrac{1}{2}\begin{pmatrix}1+\mathscr{P} & 0\\ 0 & 1-\mathscr{P}\end{pmatrix}(-\tfrac{1}{\sqrt{2}})\begin{pmatrix}e^{i\Phi}\\ ie^{i\Phi}\end{pmatrix}\right]$$
$$={}^1\!/\!_4(1+\mathscr{P}+1-\mathscr{P})=\tfrac{1}{2} \tag{51}$$

We can find the other diagonal element

$$\langle\lambda=-1|\rho|\lambda=-1\rangle=(\tfrac{1}{\sqrt{2}})(e^{i\Phi},ie^{i\Phi})\tfrac{1}{2}\begin{pmatrix}1+\mathscr{P} & 0\\ 0 & 1-\mathscr{P}\end{pmatrix}(\tfrac{1}{\sqrt{2}})\begin{pmatrix}e^{-i\Phi}\\ -ie^{-i\Phi}\end{pmatrix}$$
$$={}^1\!/\!_4(1+\mathscr{P}+1-\mathscr{P})=\tfrac{1}{2} \tag{52}$$

The off-diagonal terms are:

$$\langle\lambda=-1|\rho|\lambda=+1\rangle=-(\tfrac{1}{\sqrt{2}})(e^{i\Phi},ie^{i\Phi})\tfrac{1}{2}\begin{pmatrix}1+\mathscr{P} & 0\\ 0 & 1-\mathscr{P}\end{pmatrix}(\tfrac{1}{\sqrt{2}})\begin{pmatrix}e^{i\Phi}\\ ie^{i\Phi}\end{pmatrix}$$
$$=-{}^1\!/\!_4\left(e^{2i\Phi}(1+\mathscr{P})-e^{2i\Phi}(1-\mathscr{P})\right)$$
$$=-{}^1\!/\!_4e^{2i\Phi}\left(1+\mathscr{P}-1+\mathscr{P}\right)=-(\mathscr{P}/2)e^{2i\Phi} \tag{53}$$

and

$$\langle\lambda=+1|\rho|\lambda=-1\rangle=-(\tfrac{1}{\sqrt{2}})(e^{-i\Phi},-ie^{-i\Phi})\tfrac{1}{2}\begin{pmatrix}1+\mathscr{P} & 0\\ 0 & 1-\mathscr{P}\end{pmatrix}(\tfrac{1}{\sqrt{2}})\begin{pmatrix}e^{-i\Phi}\\ -ie^{-i\Phi}\end{pmatrix}$$
$$=-{}^1\!/\!_4\left(e^{-2i\Phi}(1+\mathscr{P})-e^{-2i\Phi}(1-\mathscr{P})\right)$$
$$=-{}^1\!/\!_4e^{-2i\Phi}\left(1+\mathscr{P}-1+\mathscr{P}\right)=-(\mathscr{P}/2)e^{-2i\Phi} \tag{54}$$

The spin density matrix of the photon in the helicity basis (and the GJ frame) is then

$$\rho_{+,-}\left(\mathscr{P},\Phi\right)=\begin{pmatrix}\langle+|\rho|+\rangle & \langle+|\rho|-\rangle\\ \langle-|\rho|+\rangle & \langle-|\rho|-\rangle\end{pmatrix}. \tag{55}$$

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$$\rho_{\lambda\lambda'}\left(\mathscr{P},\Phi\right)={}^1\!/\!_2\begin{pmatrix}1 & -\mathscr{P}e^{-2i\Phi}\\ -\mathscr{P}e^{2i\Phi} & 1\end{pmatrix} \tag{56}$$

(in agreement with reference [2]). If we compare this matrix with the expression:

$$\rho=\frac{1}{2}\begin{pmatrix}1+\mathscr{P}_3 & \mathscr{P}_1-i\mathscr{P}_2\\ \mathscr{P}_1+i\mathscr{P}_2 & 1-\mathscr{P}_3\end{pmatrix} \tag{57}$$

We obtain $\mathscr{P}_3 = 0$ and:

$$-\mathscr{P} e^{-2i\Phi} = \mathscr{P}_1 - i\mathscr{P}_2 \tag{58}$$

$$-\mathscr{P}(\cos 2\Phi - i\sin 2\Phi) = \mathscr{P}_1 - i\mathscr{P}_2 \tag{59}$$

Therefore $\mathscr{P}_1 = -\mathscr{P}\cos 2\Phi$ and $\mathscr{P}_2 = -\mathscr{P}\sin 2\Phi$, in the helicity basis:

$$\overrightarrow{\mathscr{P}} = \mathscr{P}(-\cos 2\Phi, -\sin 2\Phi, 0) \tag{60}$$

To calculate the spin density matrix in the reflectivity basis, we turn to the relations of the reflectivity basis with the helicity and the canonical bases [3].
We have that

$$|\epsilon a\lambda\rangle = \Big[|a\lambda\rangle - \epsilon P(-1)^{j-\lambda}|a-\lambda\rangle\Big]\,\Theta(\lambda) \tag{61}$$

where P is the parity of particle "a", and

$$\Theta(\lambda) = \frac{1}{\sqrt{2}} \text{ for } \lambda > 0 \tag{62}$$

$$\Theta(\lambda) = \frac{1}{2} \text{ for } \lambda = 0 \tag{63}$$

$$\Theta(\lambda) = 0 \quad \text{for } \lambda < 0 \tag{64}$$

the eigenvalue of reflectivity for λ=0 is $P(-1)^J$.
For a real photon $P = -1$, $J = 1$ and $\lambda = +1, -1$,therefore

$$|\epsilon\lambda\rangle = \frac{1}{\sqrt{2}}\Big[|\lambda\rangle - \epsilon(-1)^{\lambda}|-\lambda\rangle\Big] \tag{65}$$

then (the reflectivity eigenvalues for a photon are $\epsilon = \pm 1$).

$$\begin{aligned} |\epsilon = +1, \lambda = +1\rangle &= \tfrac{1}{\sqrt{2}}(|\lambda = +1\rangle + |\lambda = -1\rangle) \\ |\epsilon = -1, \lambda = +1\rangle &= \tfrac{1}{\sqrt{2}}(|\lambda = +1\rangle - |\lambda = -1\rangle) \end{aligned} . \tag{66}$$

Using the relations in (46), we obtain

$$\begin{aligned} |\epsilon = +1\rangle &= \tfrac{1}{\sqrt{2}}\Big[-\tfrac{1}{\sqrt{2}}|x\rangle - \tfrac{1}{\sqrt{2}}i|y\rangle + \tfrac{1}{\sqrt{2}}|x\rangle - \tfrac{1}{\sqrt{2}}i|y\rangle\Big] = -i|y\rangle \\ |\epsilon = -1\rangle &= \tfrac{1}{\sqrt{2}}\Big[-\tfrac{1}{\sqrt{2}}|x\rangle - \tfrac{1}{\sqrt{2}}i|y\rangle - \tfrac{1}{\sqrt{2}}|x\rangle + \tfrac{1}{\sqrt{2}}i|y\rangle\Big] = -|x\rangle \end{aligned} \tag{67}$$

Therefore, we find that, using values in (45)

$$\begin{aligned} \langle\epsilon = -1|\rho|\epsilon = -1\rangle &= \langle x|\rho|x\rangle = {}^1\!/_2\,(1 + \mathscr{P}\cos 2\Phi) \\ \langle\epsilon = +1|\rho|\epsilon = +1\rangle &= (i\langle y|)\rho(-i|y\rangle) = \langle y|\rho|y\rangle = {}^1\!/_2\,(1 - \mathscr{P}\cos 2\Phi) \\ \langle\epsilon = +1|\rho|\epsilon = -1\rangle &= (i\langle y|)\rho(-|x\rangle) = -i\langle y|\rho|x\rangle = -{}^1\!/_2 i\,(\mathscr{P}\sin 2\Phi) \\ \langle\epsilon = -1|\rho|\epsilon = +1\rangle &= (-\langle x|)\rho(-i|y\rangle) = i\langle x|\rho|y\rangle = {}^1\!/_2 i\,(\mathscr{P}\sin 2\Phi) \end{aligned} \tag{68}$$

We obtain the spin density matrix of the photon in the reflectivity basis:

$$\rho_{\epsilon\epsilon'}(\mathscr{P},\Phi) = 1/2 \begin{pmatrix} 1-\mathscr{P}\cos 2\Phi & -i\mathscr{P}\sin 2\Phi \\ i\mathscr{P}\sin 2\Phi & 1+\mathscr{P}\cos 2\Phi \end{pmatrix}. \tag{69}$$

If we compare this matrix with the expression:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+\mathscr{P}_3 & \mathscr{P}_1 - i\mathscr{P}_2 \\ \mathscr{P}_1 + i\mathscr{P}_2 & 1-\mathscr{P}_3 \end{pmatrix} \tag{70}$$

We obtain $\mathscr{P}_1 = 0$ and $\mathscr{P}_2 = \mathscr{P}\sin 2\Phi$ and $\mathscr{P}_3 = -\mathscr{P}\cos 2\Phi$, therefore in the reflectivity basis:

$$\overrightarrow{\mathscr{P}} = \mathscr{P}(0, \sin 2\Phi, -\cos 2\Phi) \tag{71}$$

References

[1] C.W.Salgado and D.W.Weygand; Phys.Rept. 537, 1-58 (2014).

[2] K. Schilling, P. Seyboth and G. Wolf; Nucl.Phys. **B15** ,397 (1970).

[3] S.U. Chung and T.L. Trueman; Phys.Rev.D, **11**,3, 633 (1975).