Coordinate Systems in GlueX

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Abstract

This document describes the different coordinate systems in GlueX and how to move between them. In particular, events are reconstructed in the GlueX lab frame, while results are generally presented in the center of mass frame. Studies of the decay of mesons produced in GlueX normally occur in the Gottfreid-Jackson and the Helicity frames.

1 Introduction

The goal of the GlueX Experiment [1] at Jefferson lab is to search for hybrid mesons using a linearly-polarized photon beam. Recent reviews on hybrid mesons [2, 3] provide an excellent description of the current experimental and theoretical situation. While not guaranteed to be true, it is in principle possible to produce all exotic hybridmesons in photoproduction [4].

In GlueX, we are looking at meson (M) photoproduction reactions of the form

$$\vec{\gamma}p \quad \rightarrow \quad pM\,, \tag{1}$$

where the mesonic system M subsequently decays into two or more daughter mesons. The amplitude analyses that are to be carried out in GlueX will look at the decay angular distributions of M, and that of its daughter particles. In doing these analyses, there are a number of coordinate systems that are important in the analyses. These are discussed in this note.

2 The GlueX Lab Frame

The GlueX experiment is carried out in the so-call laboratory frame, or *Lab Frame* for short. An incident photon of energy E_γ travels in the $+\hat{z}$ direction and interacts with a proton at rest in the GlueX liquid hydrogen target. In this frame, the $\hat{y}$ direction is taken as up and the $\hat{x}$ direction is taken such that we have a right-handed coordinate system:

$$\hat{x} \quad = \quad \hat{y} \times \hat{z}\,,$$

which, if looking in the $= \hat{z}$ direction, is to the left, as shown on Figure 1.

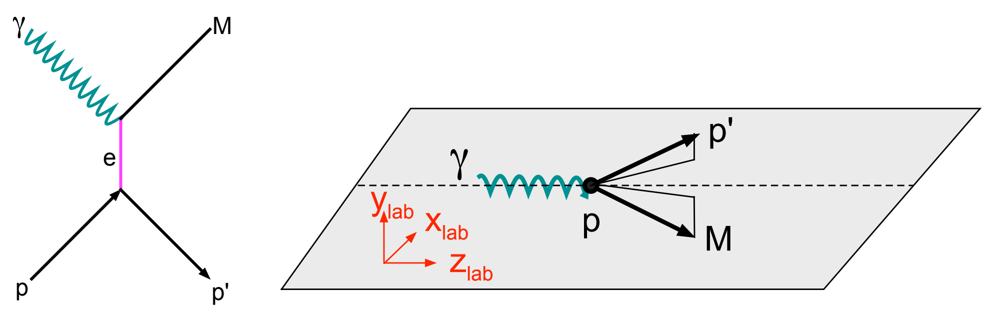

Figure 1: The Lab-Frame coordinates in Gluex. The z-axis is along the direction of the incident photon beam, the y-axis is normal to the hall floor, pointing up, and the x-axis is pointing to the left of the beam, forming a right-handed coordinate system.

In GlueX, the final-state particles are reconstructed in the lab frame, so their momentum and energy are reported in this frame. Given an initial photon of energy E_γ, we can compute the Mandelstam s as

$$s \quad = \quad \left(E_\gamma + m_pc^2\right)^2 - (E_\gamma)^2$$

$$s \quad = \quad m_pc^2\left(2E_\gamma + m_pc^2\right)\,. \tag{2}$$

3 The Center of Mass Frame

Most experimental results are reported in the center of mass frame. The frame is defined with the same axis orientations as the lab frame, but with the total momentum of the system being zero. We move from the lab frame to the center of mass frame by performing a Lorentz boost along the negative z axis. The magnitude of the β of the boost is given as

$$\beta = \frac{E_\gamma}{E_\gamma + m_p c^2},$$

and the γ factor is

$$\gamma = \frac{E_\gamma + m_p c^2}{\sqrt{m_p c^2 \left(m_p c^2 + 2E_\gamma\right)}}$$

$$\gamma = \frac{1}{\sqrt{s}} \left(E_\gamma + m_p c^2\right).$$

From this, the $\vec{\beta}$ of the boost is given as

$$\vec{\beta} = \left(0, 0, -\frac{E_\gamma}{E_\gamma + m_p c^2}\right),$$

and the Lorentz boost can be expressed as a matrix equation as in 3.

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}_{cm} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}_{lab} \quad (3)$$

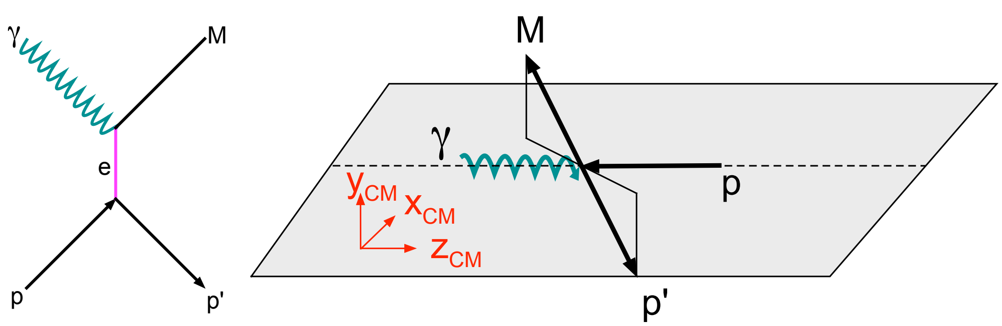

Figure 2: The GlueX center of mass frame has the same coordinate directions as the lab frame. The z-axis is along the incident photon direction, the y axis is normal to the hall floor, pointing up and the x-axis is to beam left.

4 The Gottfried-Jackson and Helicity Frames for Mesons

For reactions of the type in equation 1, the produced meson M typically decays into two (or more) daughter particles. There are two additional reference frames that we use to analyze these decays.

They are the Gottfried-Jackson (GJ) frame and the helicity ($\mathcal{H}$) frame. Both are measured in the rest frame of M, where we move to the M rest frame with a Lorentz boost from the center-of-mass frame. The frames both have the same y axis, but their z axes differ. For the GJ frame, the z-axis is taken as the direction of the incident photon in the GJ frame. Forr the $\mathcal{H}$ frame, the z-axis is taken opposite the direction of the Lorentz boost to get to the frame (along the direction of M in the center of mass).

It is important to note that one cannot just perform a Lorentz boost from the lab frame to the rest frame of M, one has to first move to the center of mass, and then to M. Unless the two Lorentz boosts are colinear, the combination of the two is the product of a pure Lorentz boost and a rotation. Thus, the direct Lorentz boost from the lab frame to the M frame will differ by the product of the two boosts by a rotation. This is known as the *Wigner rotation*, and while it is possible to compute this rotation, it is often easier just to start from the center of mass frame.

To get the Lorentz Boost to the two frames, we need the four-momentum of M (p^{μ}_{M} in the center-of-mass frame). It is of the form

$$p^{\mu}_{M} = \begin{pmatrix} E_M & p_{x_M}\,c & p_{y_M}\,c & p_{z_M}\,c \end{pmatrix},$$

where the magnitude of the three-momentum p_A is

$$p_M = \sqrt{p^2_{x_M}\,c^2 + p^2_{y_M}\,c^2 + p^2_{z_M}\,c^2}\,.$$

The magnitude of $\beta$and the relativistic factor γ are given as

$$\beta = \frac{p_M\,c}{E_M} \tag{4}$$

$$\gamma = \frac{E_M}{m_M}\,, \tag{5}$$

where m_A is the mass of particle A. We also define the β along each axis as

$$\begin{aligned} \beta_x &= \frac{p_{x_M}\,c}{E_M} \\ \beta_y &= \frac{p_{y_M}\,c}{E_M} \\ \beta_z &= \frac{p_{z_M}\,c}{E_M}\,. \end{aligned}$$

The Lorentz boost is given by the matrix equation 6.

$$\begin{pmatrix} E \\ p_x\,c \\ p_y\,c \\ p_z\,c \end{pmatrix}_{decay} = \begin{pmatrix} \gamma & -\beta_x\gamma & -\beta_y\gamma & -\beta_z\gamma \\ -\beta_x\gamma & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\beta_y\gamma & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\beta_z\gamma & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} E \\ p_x\,c \\ p_y\,c \\ p_z\,c \end{pmatrix}_{cm} \tag{6}$$

This reduces to equation 3 in the case where $\beta_x = \beta_y = 0$ and $\beta = \beta_z$.

The Helicity Frame

The Helicity frame is shown in Figure 3, which as noted above is in the rest-frame of the produced meson M. For both the $\mathcal{H}$ and the GJ frames, the $\hat{y}$-axis is defined to be the normal to the production plane. This can be expressed in terms of the center of mass momentum $\vec{p}_\gamma$ and $\vec{p}_M$ as

$$\hat{y}_{GJ,\mathcal{H}} = \frac{\vec{p}_\gamma \times \vec{p}_M}{\mid \vec{p}_\gamma \times \vec{p}_M \mid} . \tag{7}$$

In the $\mathcal{H}$ frame, the $\hat{z}_\mathcal{H}$-axis is taken opposite to the direction of the boost. in terms of the center-of-mass momentum of M, $\vec{p}_M$, this is

$$\hat{z}_\mathcal{H} = \frac{\vec{p}_M}{\mid \vec{p}_M \mid} . \tag{8}$$

Finally, the $\hat{x}_\mathcal{H}$-axis is chosen to have a right-hand coordinate system,

$$\hat{x}_\mathcal{H} = \hat{y}_\mathcal{H} \times \hat{z}_\mathcal{H} . \tag{9}$$

For the case M decaying into two or more daughter particles, the momentum of the daughter

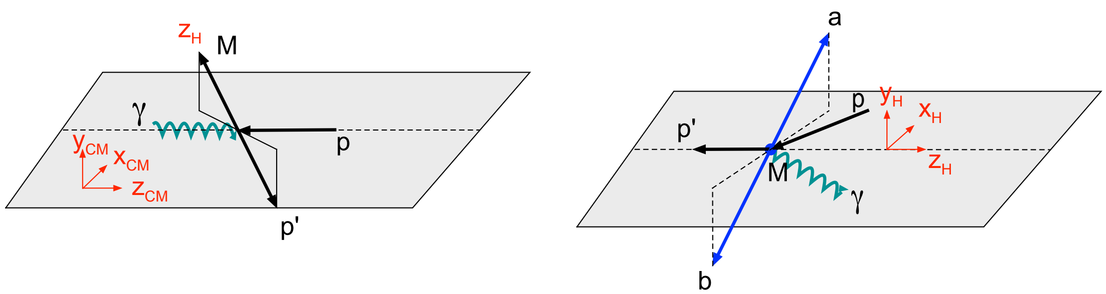

Figure 3: The Helicity Frame of the produced meson, M, (right-hand image). This is the rest frame of M with the $\hat{z}_\mathcal{H}$-axis along the direction of M in the center of mass farme, shown in the left-hand image. The $\hat{y}_\mathcal{H}$-axis is defined through equation 7 and the $\hat{x}_\mathcal{H}$-axis is taken to have a right-handed coordinate system.

particles are expressed in the $\mathcal{H}$ frame. For two daughters, a and b as shown in Figure 3, the momentum are $\vec{p}_{a_\mathcal{H}}$ and $\vec{p}_{b_\mathcal{H}}$ and define the helicity angles, $\theta_\mathcal{H}$ and $\phi_\mathcal{H}$. For particle a these are given as

$$\cos\theta_{a_\mathcal{H}} = \frac{(p_{a_\mathcal{H}})_z}{\mid \vec{p}_{a_\mathcal{H}} \mid} \tag{10}$$

$$\tan\phi_{a_\mathcal{H}} = \frac{(p_{a_\mathcal{H}})_y}{(p_{a_\mathcal{H}})_x} . \tag{11}$$

The s-Channel Frame

The s-channel frame is related to the helicity frame by a boost along the $\hat{z}_\mathcal{H}$-axis to the frame the s-channel three momentum is zero, $\vec{p}_\gamma + \vec{p}_p = 0$. This is accomplished with β the negative of equation 4 and γ given by equation 5, so the lorentz boost in terms of equations 4 and 5 is

$$\begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}.$$

The s-channel frame is just a rotated version of the center if mass frame, where the coordinate axes are those of the helicity frame. This is shown in Figure 4.

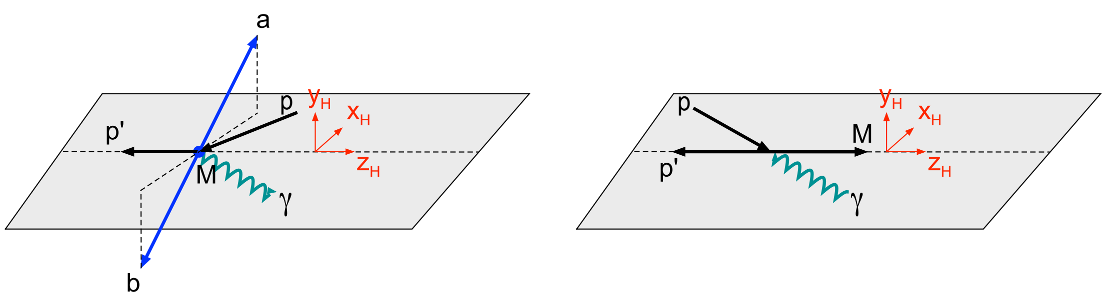

Figure 4: The s-channel frame (right) is reached from the helicity frame (left) by boosting along the $\hat{z}_{\mathcal{H}}$-axis back to the center of mass frame.

An alternate way to boost the momentum of the daughter particles of the meson M to the helicity frame is to first rotate them from the center-of-mass frame into the s-channel frame, and then use the opposite of the Lorentz boost given above.

$$\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}.$$

The Gottfried-Jacksonn Frame

For the Gottfried-Jackson frame, we start by performing the Lorentz boost from the center of mass frame to the rest frame of M as given by equation 6. In the GJ frame, the $\hat{y}_{GJ}$-axis is the same as for the helicity frame and defined by equation 7. The $\hat{z}_{GJ}$-axis is taken as the direction of the initial photon in the rest frame of the meson M. To obtain this, we need to boost four-momentum of the photon from the center of mass to the rest frame of M, also using equation 6. We then define the $\hat{z}_{GJ}$ axis as

$$\hat{z}_{GJ} = \frac{\vec{p}_\gamma(M)}{\mid \vec{p}_\gamma(M) \mid}. \tag{12}$$

Finally, the $\hat{x}_{GJ}$-axis is chosen to have a right-hand coordinate system,

$$\hat{x}_{GJ} = \hat{y}_{GJ} \times \hat{z}_{GJ}. \tag{13}$$

The GJ frame is shown in Figure 5. For the daughter particle momentum $\vec{p}_a$, we first boost them from the center of mass to the rest frame of M, $\vec{p}_a(M)$, and then use the unit vectors, equations 13, 7 and 12, to project out the components of momentum,

$$\vec{p}_a(GJ) \quad = \quad (\vec{p}_a(M) \cdot \hat{x}_{GJ} \, , \, \vec{p}_a(M) \cdot \hat{y}_{GJ} \, , \, \vec{p}_a(M) \cdot \hat{z}_{GJ}) \ .$$

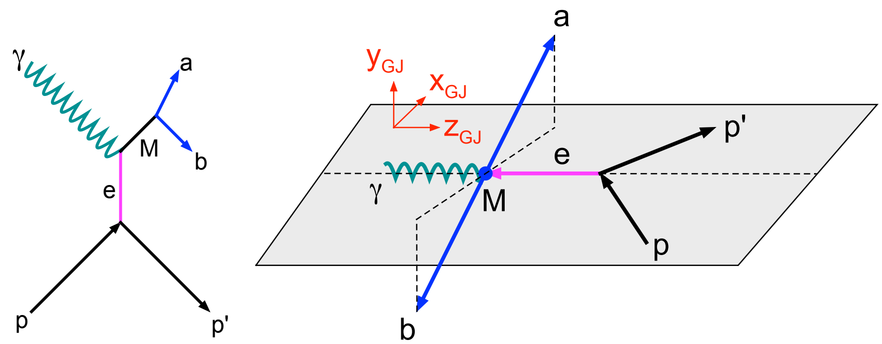

Figure 5: The Gottfried-Jackson frame for M. This is the rest-frame of M with the z-axis taken along the direction of the photon in the frame.

An alternate approach is to first move the particle's momentum to the helicity frame, and then perform a rotation about the $\hat{y}_{\mathcal{H}}$ axis by an angle α, where β is obtained from equation 4.

$$\cos\alpha \quad = \quad \frac{\cos\theta_{cm}(M) - \beta}{1 - \beta\cos\theta_{cm}(M)}$$

The t-Channel Frame

Similar to how the helicity frame and the s-channel frame are related, the Gottfried-Jackson frame is related to the t-channel frame. The t-channel frame is reached by performing a Lorentz boost along $\hat{z}_{GJ}$ from the GJ frame to a frame where the t-channel three momentum is zero, $\vec{p}_M - \vec{p}_\gamma = 0$.

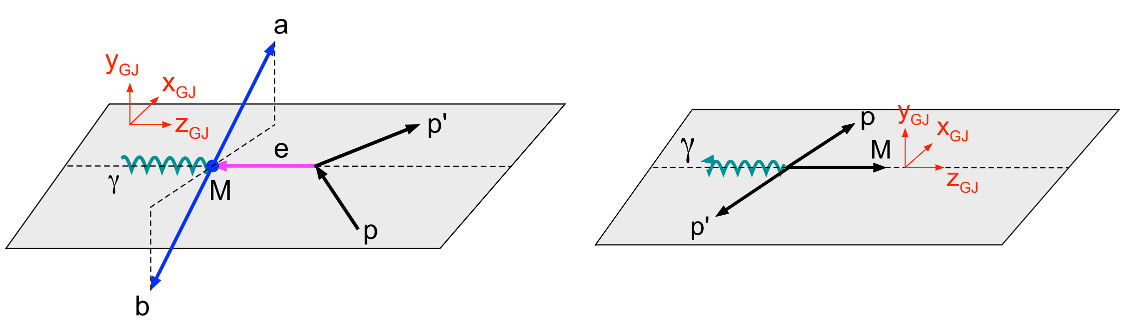

Figure 6: The t-channel frame (right) is reached from the Gottfried-Jackson frame (left) by boosting along the $\hat{z}_{GJ}$-axis to the frame where $\vec{p}_M - \vec{p}_\gamma = 0$.

5 Decays of Daughter Particles

For the photoproduction of M through reaction 1, we have considered the subsequent decay of M to daughter particles,

$$M \rightarrow ab$$

where we can look at the decay of M in either the GJ or $\mathcal{H}$ frames (of M). It often happens that one or both of the daughter particles (a and/or b) subsequently decays to grand daughter particles, for example

$$a \rightarrow cd\,.$$

In this case, we usually look at the subsequent decay (of a) in the $\mathcal{H}$ frame for a. The procedure is the same, in that we start from the rest frame of M and perform a Lorentz boost into the rest frame of a. The $\hat{y}$ axis is taken as the normal to the decay plane

$$\hat{y}_{GJ,\mathcal{H}} = \frac{\hat{z}_{\mathcal{H}} \times \vec{p}_a}{\mid \hat{z}_{\mathcal{H}} \times \vec{p}_a \mid}\,, \tag{14}$$

where $\vec{p}_a$ is measured in the rest frame of M. This formulation breaks down when the grand-daughter particles travel along the $\hat{z}_{\mathcal{H}}(M)$ axes, so care needs to be exercised in that limit. The z-axis is then defined as before, with

$$\hat{z}_{\mathcal{H}}(a) = \frac{\vec{p}_a}{\mid \vec{p}_a \mid}\,, \tag{15}$$

for the helicity frame. This procedure can continue to be extended for subsequent decays as well.

References

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[4] A. A. Meyer, **The Production and Decay of Normal and Exotic-Hybrid Mesons in GlueX**, GlueX-Doc 4788 (2020).