Partial wave analysis studies with simulated $\eta^{(')}\pi^0$ events in GlueX

Florida International University 2020

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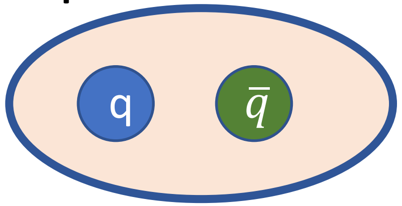
Mesons

Mesons in standard quark model



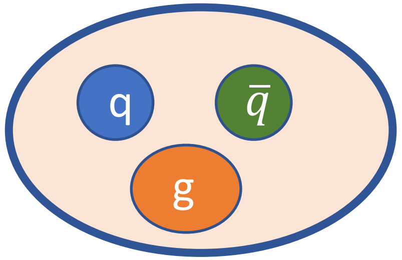
Classified as J^{PC} multilets:

$\vec{J} = \vec{L} + \vec{S}$,
$P = (-1)^{L+1} \rightarrow$ Spherical harmonics $(-1)^{l}$
$\times$ Product of individual parites of $q, \bar{q}$ (-1)
$C = (-1)^{L+S} \rightarrow$ Orbital angular momentum $(-1)^{l}$
$\times$ Flip of spin wavefunctions $(-1)^{S+1}$
$\times$ interchanging q and $\bar{q}$ (-1)

J- total angular momentum
S- total quark spin
L- orbital angular momentum between $q\bar{q}$ pair
P- parity
C- charge conjugation

J^{PC}=$\mathbf{0}^{--}$, $\mathbf{odd}^{-+}$ and $\mathbf{even}^{+-}$ **"exotic"** quantum numbers are not available.

Hybrid mesons



Quark anti-quark pair coupled to valence gluon.
"Exotic" J^{PC} are also available.
Predicted by lattice QCD (quantum chromodynamics) calculations (Phys. Rev. D 88, 094505 (2013)).

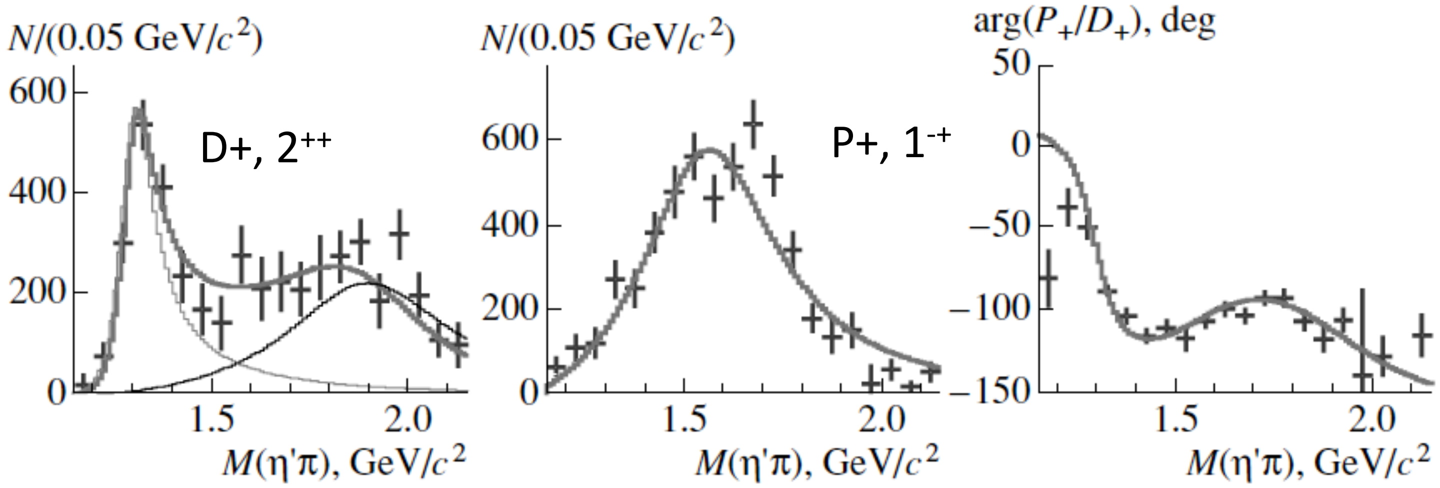
Primary motivation of the GLUEX is the search for light hybrid mesons.

π_1 (1600) results from studies of $\eta'\pi$ system with π beam incident on a p target

Evidence for exotic $I^G J^{PC} = 1^-1^{-+}$ state π_1 (1600) produced via natural parity exchange (exchanged particle with J^Ps of 0+,1-,2+…)

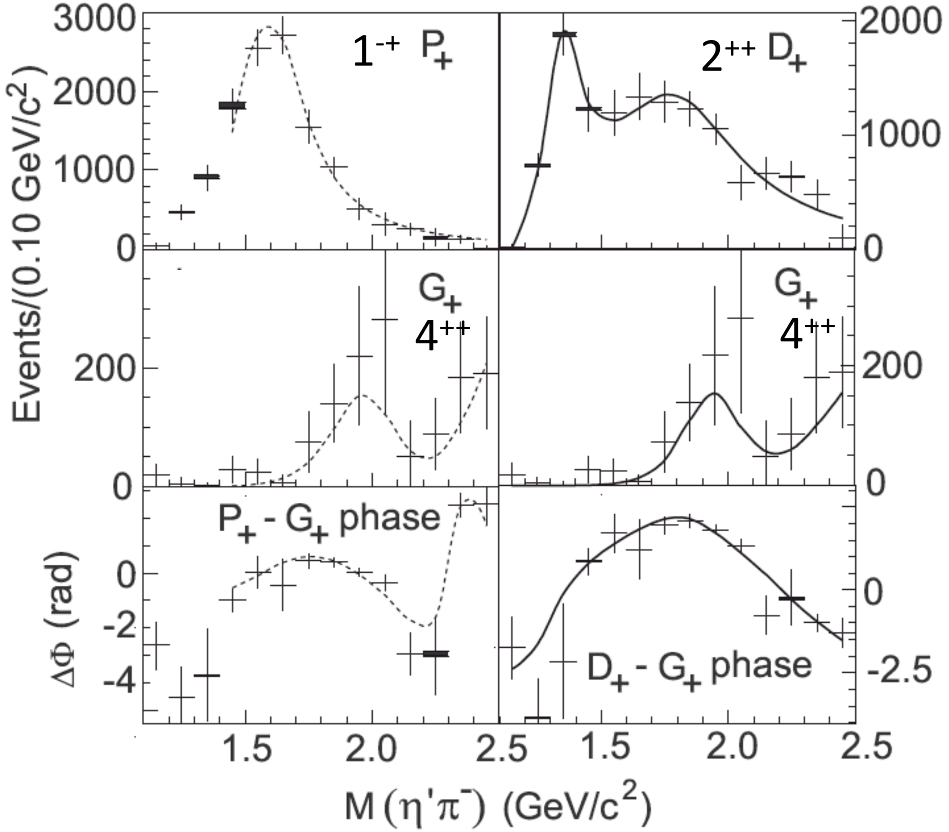
G = C · $(-1)^I$, C operator followed by a rotation in isospin (I)

VES



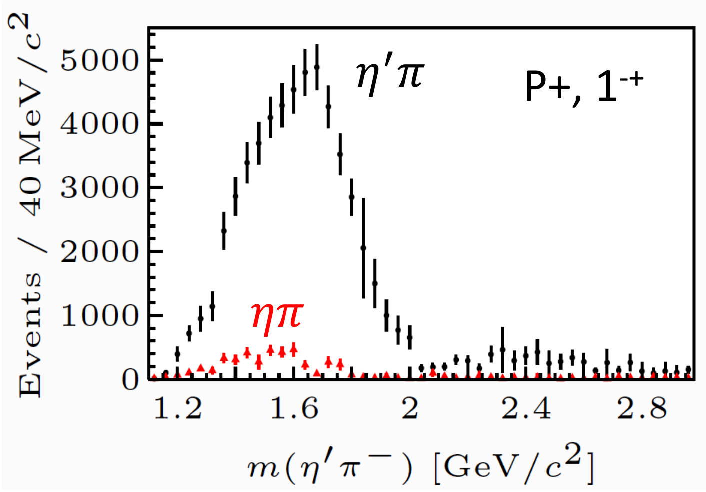
D. V. Amelin et al., Phys. Atom. Nucl. 68, 359 (2005)

E852



E. I. Ivanov et al. [E852 Collaboration], Phys. Rev. Lett. 86, 3977 (2001)

COMPASS



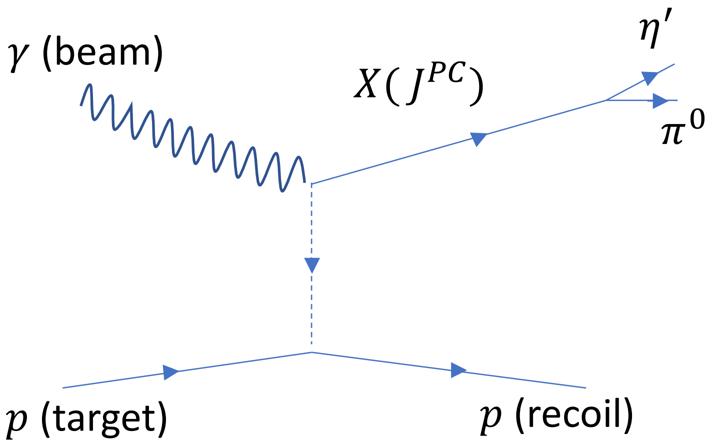
C. Adolph, et al. [COMPASS Collaboration], Phys. Lett. B740, 303 (2015)

Search for exotic π_1 (1600) using the reaction $\gamma p \to p\eta'\pi^0$ in GLUEX

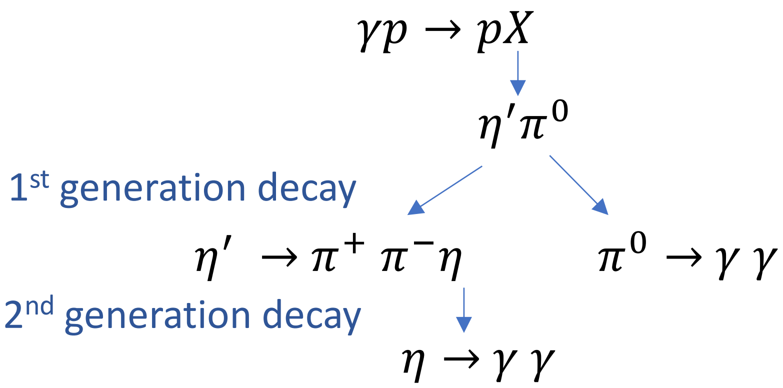
The odd waves in $\eta'\pi^0$ mesonic system have exotic quantum numbers and the lowest of them, the P-wave corresponds to exotic $\pi_1(1600)$ state.

GLUEX uses linearly polarized photon beam with E_γ~ 9GeV

$t -$ channel exchange



Decay mode



Model for Intensity with polarized photon beam

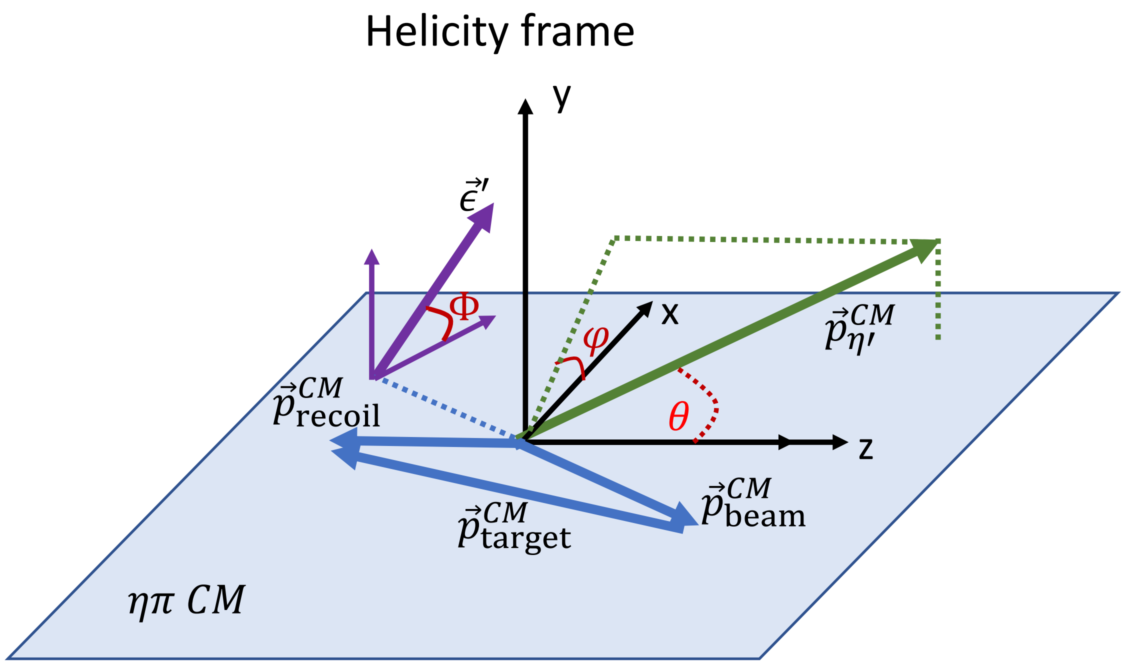
$\vec{\gamma}(\lambda, p_\gamma)p(\lambda_1, p_N) \longrightarrow \pi^0(p_\pi)\,\eta(p_\eta)p(\lambda_2, p_N')$

Φ - angle between γ polarization vector $\vec{\epsilon}'$ and production plane

Ω- direction of η in helicity frame

P_γ is the degree of linear polarization

$A_{\lambda;\lambda_1\lambda_2}(\Omega)$-the reaction amplitude



$$I(\Omega, \Phi) = \frac{d\sigma}{dt\,dm_{\eta\pi}\,d\Omega\,d\Phi}$$

$$I(\Omega, \Phi) = I^0(\Omega) - P_\gamma I^1(\Omega)\cos 2\Phi - P_\gamma I^2(\Omega)\sin 2\Phi$$

$$I^0(\Omega) = \frac{\kappa}{2}\sum_{\lambda,\lambda_1,\lambda_2} A_{\lambda;\lambda_1\lambda_2}(\Omega)\, A^*_{\lambda;\lambda_1\lambda_2}(\Omega),$$

$$I^1(\Omega) = \frac{\kappa}{2}\sum_{\lambda,\lambda_1,\lambda_2} A_{-\lambda;\lambda_1\lambda_2}(\Omega)\, A^*_{\lambda;\lambda_1\lambda_2}(\Omega),$$

$$I^2(\Omega) = i\frac{\kappa}{2}\sum_{\lambda,\lambda_1,\lambda_2} \lambda A_{-\lambda;\lambda_1\lambda_2}(\Omega)\, A^*_{\lambda;\lambda_1\lambda_2}(\Omega),$$

with κ containing all kinematical factors. The partial wave amplitudes T^l are defined by:

$$A_{\lambda;\lambda_1\lambda_2}(\Omega) = \sum_{lm} T^l_{\lambda m;\lambda_1\lambda_2} Y^m_l(\Omega)$$

We introduce reflectivity basis which allows to trade helicity λ for the reflectivity index ϵ=$\pm$1, and express helicity amplitudes in terms of reflectivity amplitudes

$$T^l_{-1m;\lambda_1\lambda_2} = (-1)^m[{}^{(-)}T^l_{-m;\lambda_1\lambda_2} - {}^{(+)}T^l_{-m;\lambda_1\lambda_2}] \qquad T^l_{+1m;\lambda_1\lambda_2} = {}^{(-)}T^l_{m;\lambda_1\lambda_2} + {}^{(+)}T^l_{m;\lambda_1\lambda_2}$$

At high energies, t-channel exchange and natural (unnatural) exchanges contributes only to the $\epsilon=+(\epsilon=-)$ components in the reflectivity basis.

Define phase rotated spherical harmonics

$$Z^m_l(\Omega, \Phi) \equiv Y^m_l(\Omega)e^{-i\Phi}$$

$$\mathrm{Re}Z^m_l(\Omega, \Phi) = \sqrt{\frac{2l+1}{4\pi}}\, d^l_{m0}(\theta)\cos(m\varphi - \Phi)$$

$$\mathrm{Im}Z^m_l(\Omega, \Phi) = \sqrt{\frac{2l+1}{4\pi}}\, d^l_{m0}(\theta)\sin(m\varphi - \Phi)$$

Mathieu et al. Phys. Rev. D 100, 054017 (2019)

Model for Intensity with polarized photon beam

Parity invariance implies

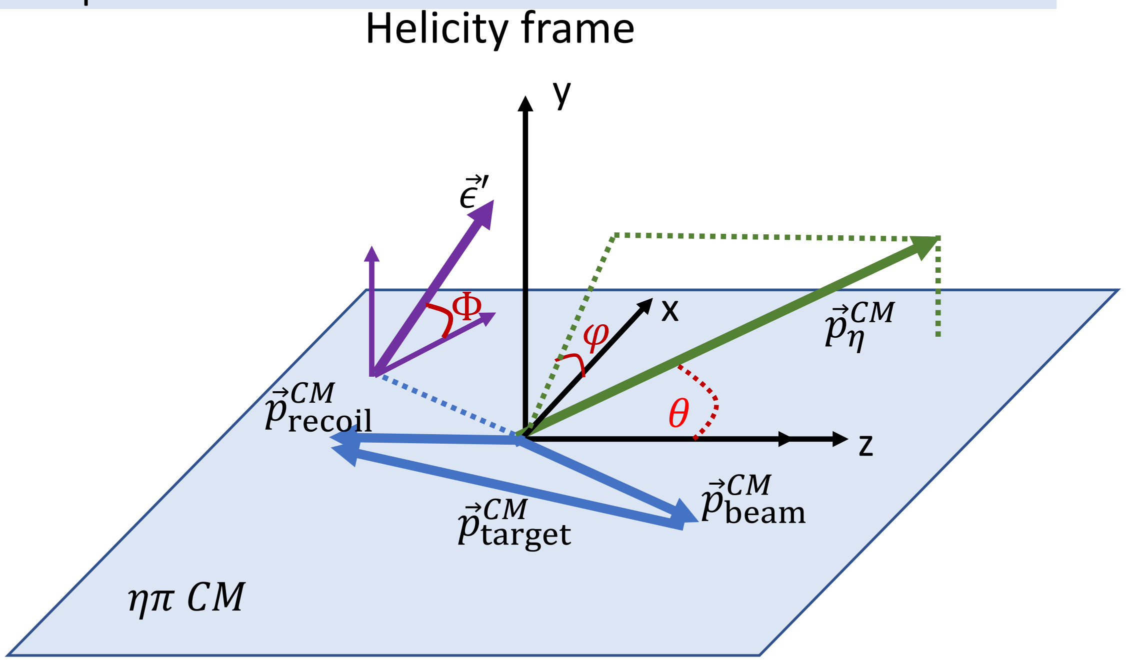
$${}^{(\epsilon)}T^{l}_{m;-\lambda_1-\lambda_2} = \epsilon(-1)^{\lambda_1-\lambda_2}\ {}^{(\epsilon)}T^{l}_{m;\lambda_1\lambda_2}$$

We take advantage of this constraint to define

$$l^{(\epsilon)}_{m;0}[l] = {}^{(\epsilon)}T^{l}_{m;++}\quad l^{(\epsilon)}_{m;1}[l] = {}^{(\epsilon)}T^{l}_{m;+-}$$

Are partial wave amplitudes for spin flip k=1 and spin non-flip k=0.
For each l , there are 2*2*(2l+1) complex partial waves with $\epsilon=\pm1$, k=0,1 corresponding to target and recoil helicities and m=-l,…..l.
There is no interference between $\epsilon=+$and $\epsilon=-$ intensities.

Intensity that involves four coherent sums for each configuration of nucleon spin:



$$I(\Omega,\Phi) = 2\kappa\sum_{k}\left\{(1-P_\gamma)\left|\sum_{l,m}[l]^{(-)}_{m;k}\mathrm{Re}[Z^m_l(\Omega,\Phi)]\right|^2 + (1-P_\gamma)\left|\sum_{l,m}[l]^{(+)}_{m;k}\mathrm{Im}[Z^m_l(\Omega,\Phi)]\right|^2 + (1+P_\gamma)\left|\sum_{l,m}[l]^{(+)}_{m;k}\mathrm{Re}[Z^m_l(\Omega,\Phi)]\right|^2 + (1+P_\gamma)\left|\sum_{l,m}[l]^{(-)}_{m;k}\mathrm{Im}[Z^m_l(\Omega,\Phi)]\right|^2\right\}$$

Helicity-non-flip amplitudes dominate and we set the helicity-flip amplitudes to zero. This is not restrictive as the target is not polarized in GlueX, and the measured intensities are not sensitive to the details of the nucleon helicity structure.
Natural parity exchanges (corresponding to the amplitudes with $\epsilon=+1$) dominate in the energy range of interest.

- Bin data in small bins of $m_{\eta\pi}$, t and E_γ with constant $V_{\epsilon LM}$
- Fit data using extended unbinned (in (θ,φ)) maximum likelihood method

$$\ln L(V) = \sum_{i=1}^{N}\ln I(V,\theta,\varphi) - \int I(V,\theta,\varphi)\eta(\theta,\varphi)\,d\Omega$$

$\eta(\theta,\varphi)$ -acceptance

- Minimize –lnL using MINUIT, to find V

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Generated $2*10^6$ $(p\eta\pi^0)$events with AmpTools

Generated resonances are

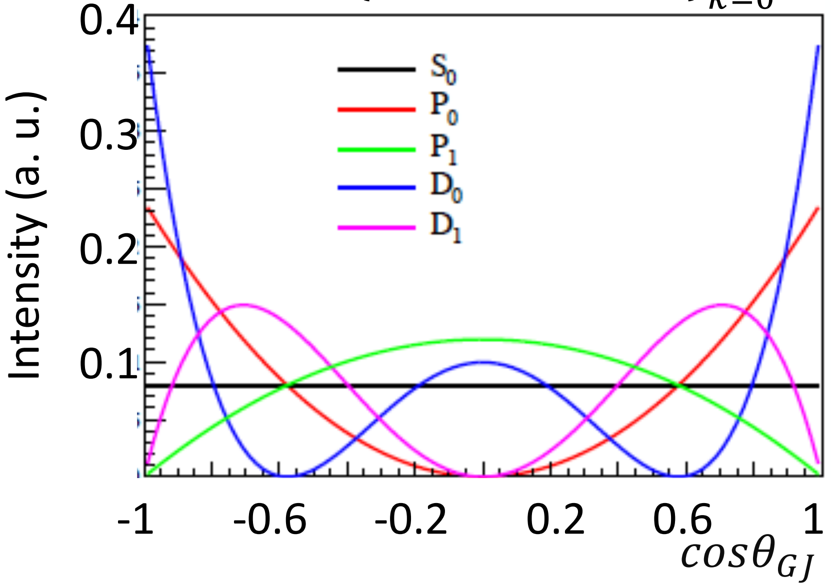
- a_0(980 MeV)
- π_1(1600 MeV) (exotic)
- a2 (1320 MeV)
- a2’ (1700 MeV)

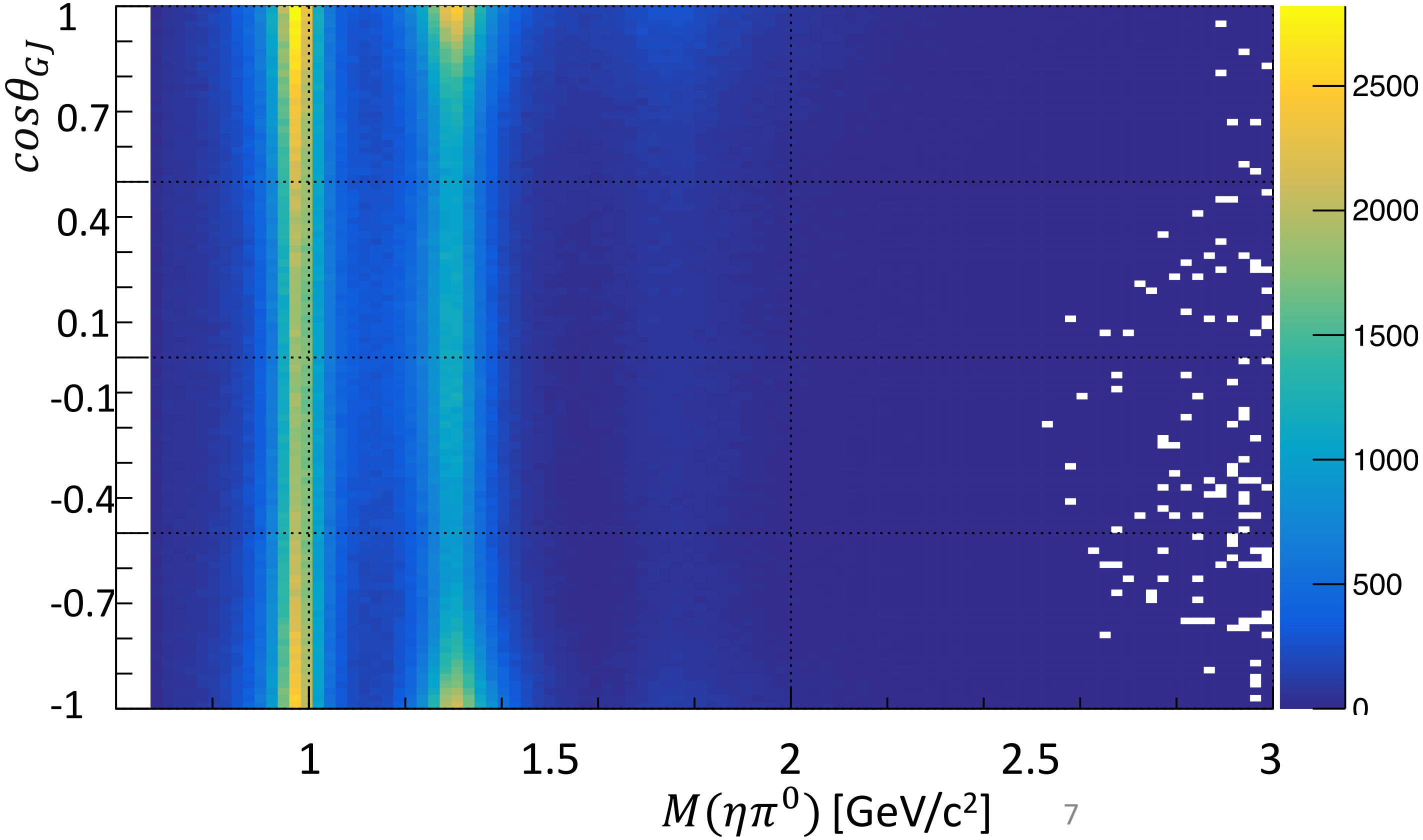
θ_{pol}=1.77 Deg.

P_γ =0.3

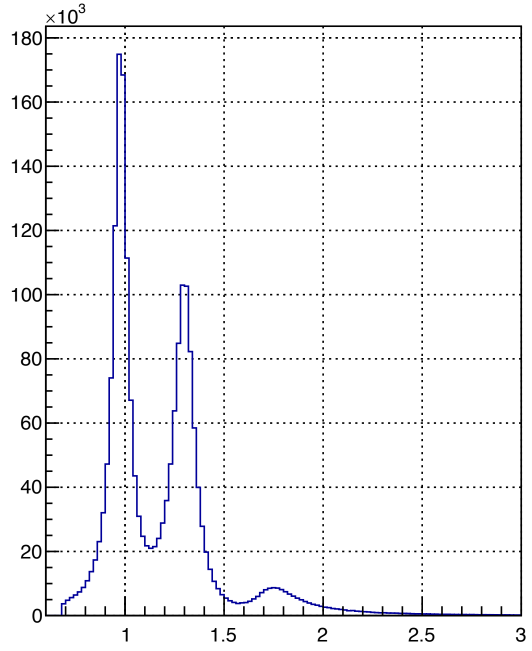
J	M	ε	Real	Imaginary	BW Mass	BW Width
0	0	+1	1000	0	0.980	0.075
1	0, 1	+1	70	70	1.564	0.492
2	0,1,2	+1	150	150	1.306	0.114
2	0,1,2	+1	50	50	1.722	0.247

The wave set: $[l]^{(\epsilon)}_{m;k} = \left\{S_0^{(+)}, P_{0,1}^{(+)}, D_{0,1,2}^{(+)}\right\}_{k=0}$ with M>= 0

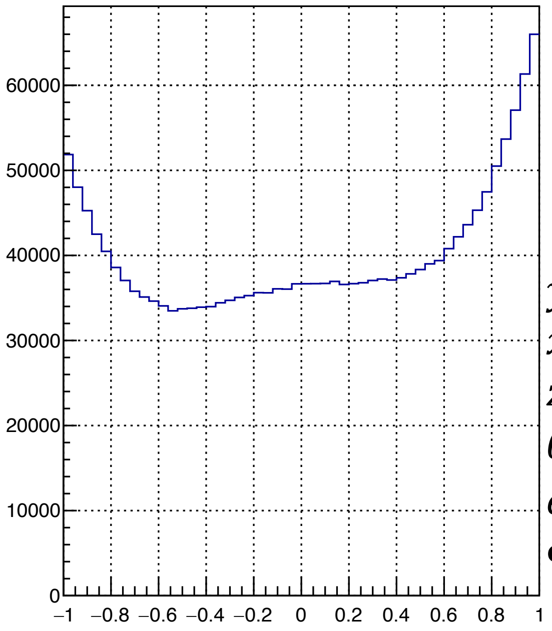




Generated $2*10^6$ $(p\eta'\pi^0)$ events with AmpTools

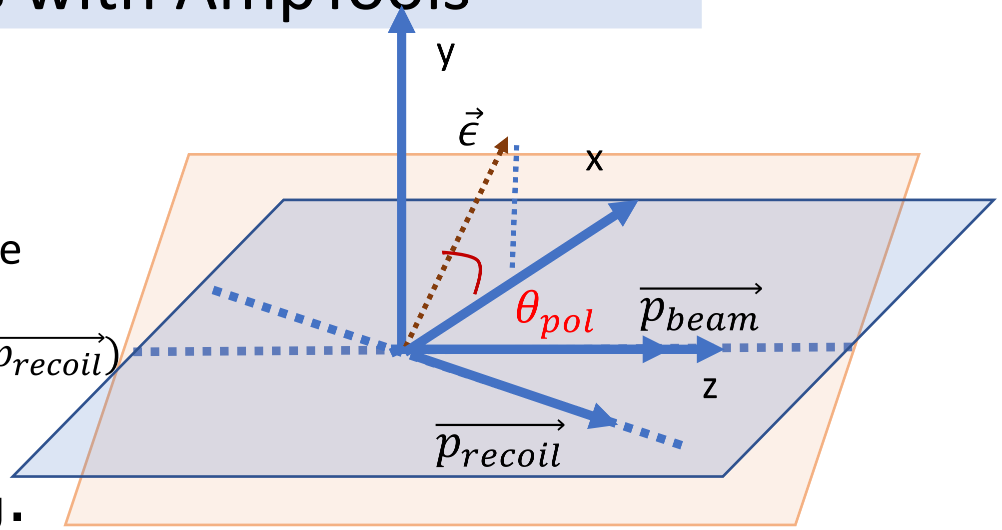


$M(\eta'\pi^0)$ [GeV/c²]



$\cos\theta_{GJ}$

Lab frame



$\vec{y} = \overrightarrow{p_{beam}} \times (-\overrightarrow{p_{recoil}})$

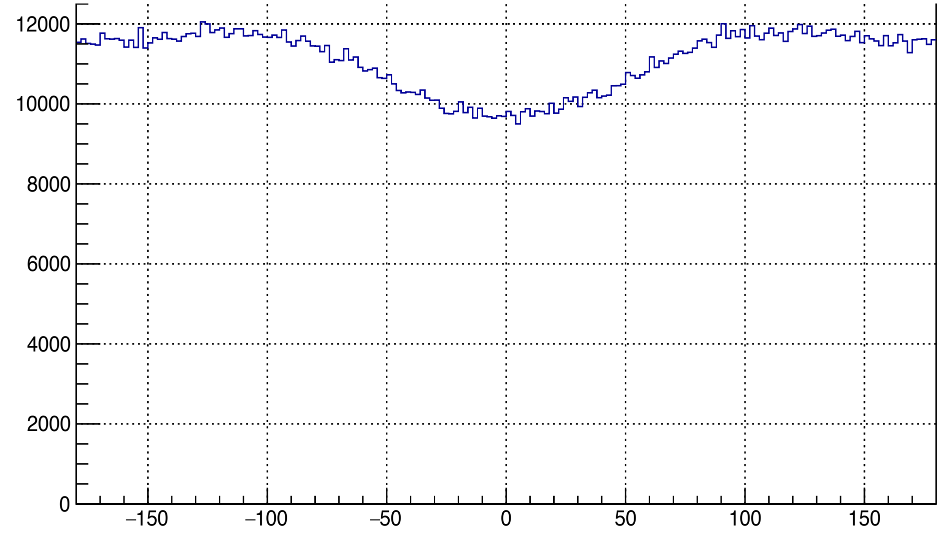
$\vec{x} = \vec{y} \times \overrightarrow{p_{beam}}$

$\vec{z} = \vec{x} \times \vec{y}$

θ_{pol}=1.7 Deg.

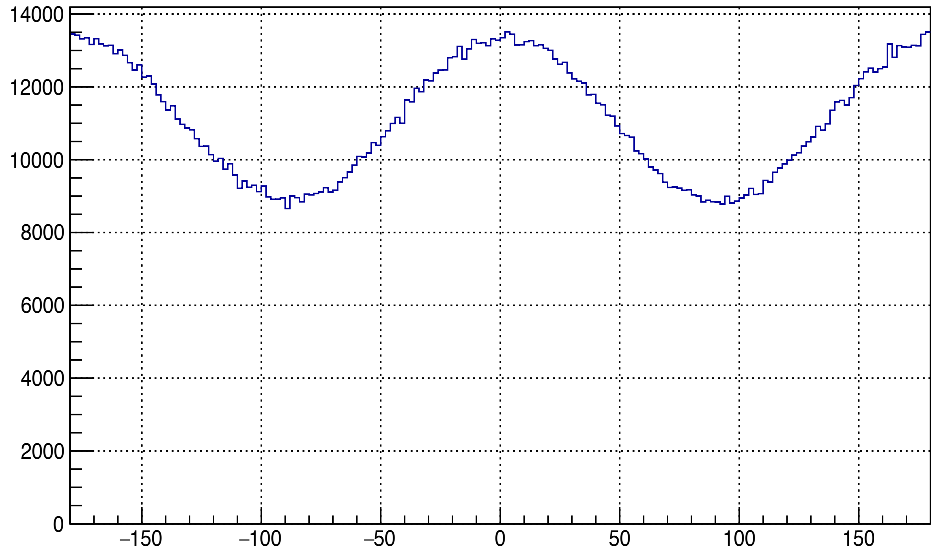
$\vec{\epsilon} = (\cos(\theta_{pol}),\ \sin(\theta_{pol}),\ 0)$

$\Phi = \mathrm{arctg}(\vec{y}\cdot\vec{\epsilon},\ \overrightarrow{p_{beam}}\cdot(\vec{\epsilon}\times\vec{y}))$



φ

Lab frame



Φ

Analysis strategy

For the wave set $[l]^{(\epsilon)}_{m;k} = \left\{S_0^{(+)}, P_{0,1}^{(+)}, D_{0,1,2}^{(+)}\right\}_{k=0}$ with M>= 0

1. Fit intensity to find partial waves using AmpTools.

2. Implement and test calculation of moments in terms of partial waves using the following expressions in terms of the $\eta'\pi^0$ SDMEs calculated in reflectivity basis:

$$H^0(LM) = \sum_{\substack{\ell\ell' \\ mm'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} C^{\ell 0}_{\ell' 0 L 0} C^{\ell m}_{\ell' m' L M}\, \rho^{\alpha,\ell\ell'}_{mm'}$$

$$H(LM) = -\sum_{\substack{\ell\ell' \\ mm'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} C^{\ell 0}_{\ell' 0 L 0} C^{\ell m}_{\ell' m' L M}\, \rho^{\ell\ell'}_{mm'}$$

$$\rho^{\alpha,ll'}_{mm'} = \sum_{\epsilon} {}^{(\epsilon)}\rho^{\alpha,ll'}_{mm'}$$

$${}^{(\epsilon)}\rho^{0,\ell\ell'}_{mm'} = \kappa \sum_k \left([\ell]^{(\epsilon)}_{m;k}[\ell']^{(\epsilon)*}_{m';k} + (-1)^{m-m'}[\ell]^{(\epsilon)}_{-m;k}[\ell']^{(\epsilon)*}_{-m';k}\right),$$

$${}^{(\epsilon)}\rho^{1,\ell\ell'}_{mm'} = -\epsilon\kappa \sum_k \left((-1)^m[\ell]^{(\epsilon)}_{-m;k}[\ell']^{(\epsilon)*}_{m';k} + (-1)^{m'}[\ell]^{(\epsilon)}_{m;k}[\ell']^{(\epsilon)*}_{-m';k}\right),$$

$${}^{(\epsilon)}\rho^{2,\ell\ell'}_{mm'} = -i\epsilon\kappa \sum_k \left((-1)^m[\ell]^{(\epsilon)}_{-m;k}[\ell']^{(\epsilon)*}_{m';k} - (-1)^{m'}[\ell]^{(\epsilon)}_{m;k}[\ell']^{(\epsilon)*}_{-m';k}\right),$$

$${}^{(\epsilon)}\rho^{3,\ell\ell'}_{mm'} = \kappa \sum_k \left([\ell]^{(\epsilon)}_{m;k}[\ell']^{(\epsilon)*}_{m';k} - (-1)^{m-m'}[\ell]^{(\epsilon)}_{-m;k}[\ell']^{(\epsilon)*}_{-m';k}\right).$$

3. Compare the results from previous step to moment distributions obtained by Monte Carlo integrations based on the expressions:

$$H^0(LM) = \frac{P_\gamma}{2}\int_\circ I(\Omega,\Phi)\, d^L_{M0}(\theta)\cos M\phi,$$

$$H^1(LM) = \int_\circ I(\Omega,\Phi)\, d^L_{M0}(\theta)\cos M\phi \,\cos 2\Phi,$$

$$\mathrm{Im}\, H^2(LM) = -\int_\circ I(\Omega,\Phi)\, d^L_{M0}(\theta)\sin M\phi\,\sin 2\Phi,$$

with $\int_\circ = (1/\pi P_\gamma)\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\Phi$

4. Compare moments from fitting with true wave set (S0+, P0+,P1+,D0+,D1+,D2+) using good starting values for fit parameters (partial wave amplitudes $[l]^{(\epsilon)}_{m;k}$) to moments from:

- Fit 1 : fitting with S0+, P0+, D0+, D1+, G0+ ,G1+ waveset using good starting values for the fit parameters that are common
- Fit 2: fitting with S0-, P0+, P1+, D0+, D1+, D2+, G0+ ,G1+ waveset using good starting values for the fit parameters that are common

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Implementation of calculation of moments

1. Implement and test calculation of moments in terms of partial waves using the following expressions in terms of the $\eta'\pi^0$ SDMEs calculated in reflectivity basis:

$$H^0(LM) = \sum_{\substack{\ell\ell' \\ mm'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} C^{\ell 0}_{\ell' 0 L 0} C^{\ell m}_{\ell' m' L M}\, \rho^{\alpha,\ell\ell'}_{mm'} \qquad \rho^{\alpha,ll'}_{mm'} = \sum_{\epsilon} {}^{(\epsilon)}\rho^{\alpha,ll'}_{mm'}$$

$$H(LM) = -\sum_{\substack{\ell\ell' \\ mm'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} C^{\ell 0}_{\ell' 0 L 0} C^{\ell m}_{\ell' m' L M}\, \rho^{\ell\ell'}_{mm'}$$

$$ {}^{(\epsilon)}\rho^{0,\ell\ell'}_{mm'} = \kappa \sum_k \left([\ell]^{(\epsilon)}_{m;k} [\ell']^{(\epsilon)*}_{m';k} + (-1)^{m-m'} [\ell]^{(\epsilon)}_{-m;k} [\ell']^{(\epsilon)*}_{-m';k} \right),$$

$$ {}^{(\epsilon)}\rho^{1,\ell\ell'}_{mm'} = -\epsilon\kappa \sum_k \left((-1)^{m} [\ell]^{(\epsilon)}_{-m;k} [\ell']^{(\epsilon)*}_{m';k} + (-1)^{m'} [\ell]^{(\epsilon)}_{m;k} [\ell']^{(\epsilon)*}_{-m';k} \right),$$

$$ {}^{(\epsilon)}\rho^{2,\ell\ell'}_{mm'} = -i\epsilon\kappa \sum_k \left((-1)^{m} [\ell]^{(\epsilon)}_{-m;k} [\ell']^{(\epsilon)*}_{m';k} - (-1)^{m'} [\ell]^{(\epsilon)}_{m;k} [\ell']^{(\epsilon)*}_{-m';k} \right),$$

$$ {}^{(\epsilon)}\rho^{3,\ell\ell'}_{mm'} = \kappa \sum_k \left([\ell]^{(\epsilon)}_{m;k} [\ell']^{(\epsilon)*}_{m';k} - (-1)^{m-m'} [\ell]^{(\epsilon)}_{-m;k} [\ell']^{(\epsilon)*}_{-m';k} \right).$$

The calculation is implemented by me in "project_moments_polarized"
Another version of the calculation based on Vincents codes is called "Pol_moments_viafittedPW"

2. The calculation based on explicit formulas

$$H^0(00) = H^1(00) + 2\left[|P_1^{(+)}|^2 + |D_1^{(+)}|^2 + |D_2^{(+)}|^2\right]$$

$$H^1(22) = H^0(22) + \frac{\sqrt{6}}{7}|D_1^{(+)}|^2 + \frac{\sqrt{6}}{5}|P_1^{(+)}|^2$$

that is applicable for M>=0, ϵ>0 and L<=D is coded in "project_moments_SPD_etapi0_posepsilon".
All three codes can be found in halld_sim/src/programs/AmplitudeAnalysis/

3. I have also added scripts and codes for plotting moments in hd_utilities/PWA_scripts/Polarized_moments_viaPW

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Config file for fitting with generated amplitudes in M and t bins

```
define polVal 0.3
fit FITNAME
reaction EtaPrimePi0 Beam Proton Eta Pi0

genmc  EtaPrimePi0 ROOTDataReader GENMCFILE
accmc  EtaPrimePi0 ROOTDataReader ACCMCFILE
data EtaPrimePi0 ROOTDataReader DATAFILE

sum EtaPrimePi0 PositiveRe
sum EtaPrimePi0 PositiveIm
parameter polAngle 1.77 fixed

# a0(980)
amplitude EtaPrimePi0::PositiveIm::S0+ Zlm 0 0 -1 -1 polAngle polVal
amplitude EtaPrimePi0::PositiveRe::S0+ Zlm 0 0 +1 +1 polAngle polVal
# a2(1320)a2'(1700)
amplitude EtaPrimePi0::PositiveIm::D0+ Zlm 2 0 -1 -1 polAngle polVal
amplitude EtaPrimePi0::PositiveRe::D0+ Zlm 2 0 +1 +1 polAngle polVal
amplitude EtaPrimePi0::PositiveIm::D1+ Zlm 2 1 -1 -1 polAngle polVal
amplitude EtaPrimePi0::PositiveRe::D1+ Zlm 2 1 +1 +1 polAngle polVal
amplitude EtaPrimePi0::PositiveIm::D2+ Zlm 2 2 -1 -1 polAngle polVal
amplitude EtaPrimePi0::PositiveRe::D2+ Zlm 2 2 +1 +1 polAngle polVal
# pi1(1600)        • • •

initialize EtaPrimePi0::PositiveIm::S0+ cartesian 1000.0 0.0 real
initialize EtaPrimePi0::PositiveRe::S0+ cartesian 1000.0 0.0 real
initialize EtaPrimePi0::PositiveIm::D0+ cartesian 70.0 70.0
initialize EtaPrimePi0::PositiveRe::D0+ cartesian 70.0 70.0
initialize EtaPrimePi0::PositiveIm::D1+ cartesian 70.0 70.0
initialize EtaPrimePi0::PositiveRe::D1+ cartesian 70.0 70.0
initialize EtaPrimePi0::PositiveIm::D2+ cartesian 70.0 70.0
initialize EtaPrimePi0::PositiveRe::D2+ cartesian 70.0 70.0
    • • •
constrain EtaPrimePi0::PositiveIm::S0+ EtaPrimePi0::PositiveRe::S0+
    • • •
```

Typically refers to unique set of initial and final state particles
Can also refer to multiple decay modes of the same set of final state particles

Reaction, data reader class, argument

Events to fit intensity to

All amplitudes within a given sum are added coherently

User defined classes

Keywords

Reaction, Sum, amplitude name, amplitude class, arguments

```
Zlm as suggested in GlueX doc-4094 (M. Shepherd)
argument 1 : j
argument 2 : m
argument 3 : real (+1) or imaginary (-1) part
argument 4 : 1 + (+1/-1) * P_gamma
argument 5 : polarization angle (in Deg.)
argument 6 : beam properties config file or fixed polarization
```

Initial value of partial wave amplitudes $[l]^{(\epsilon)}_{m;k}$ in cartesian coordinate system

Factors with the same reaction sum and amplitude name are multiplied together

Same amplitudes corresponding to different sums should be equal

Results for bin M=1.37 and t<0.3

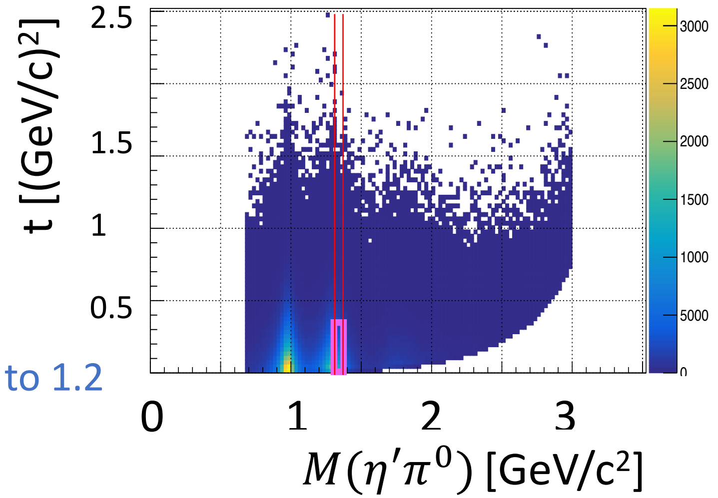
Amplitudes used in fitting are S0+, P0+, P1+, D0+, D1+, D2+. Good starting values for fit parameters

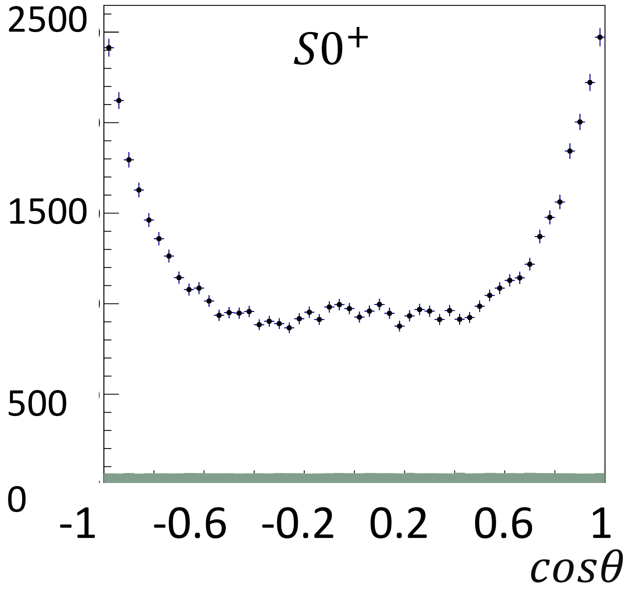
Fit results

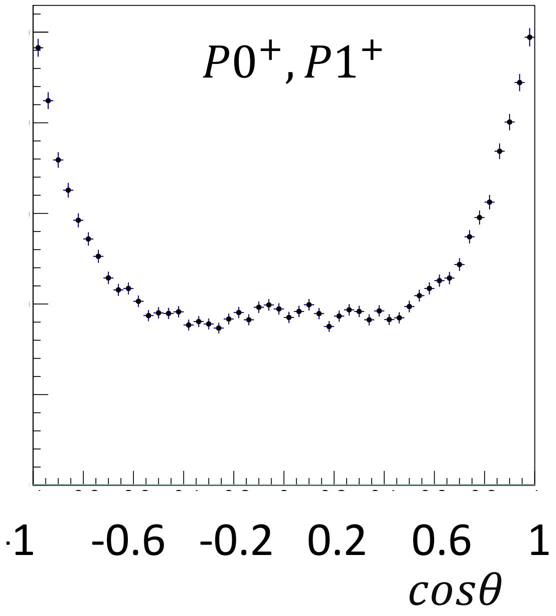
Bin M , t

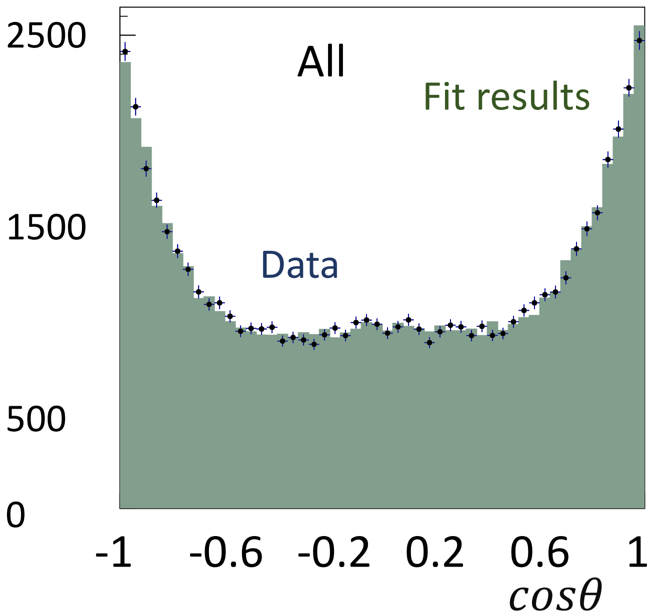
$M(\eta\pi^0)$ range from 0.7 to 3
N bins=45
Bin width≈ 0.051 GeV/c^2

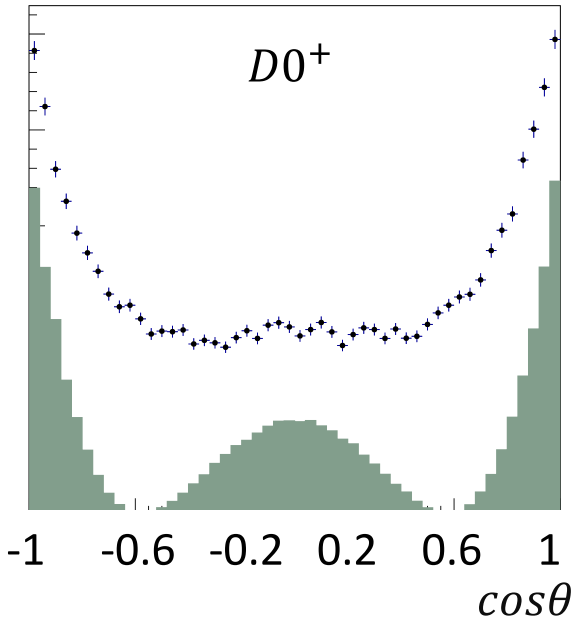
t range from 0 to 1.2
N bins=4
Bin width≈ 0.3 $(GeV/c)^2$

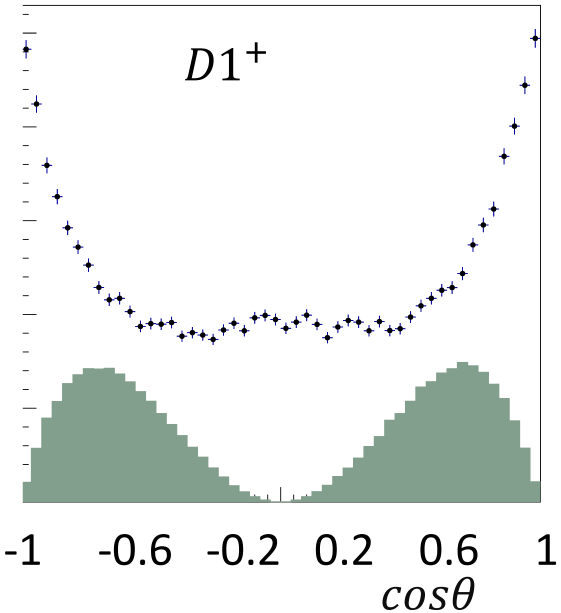


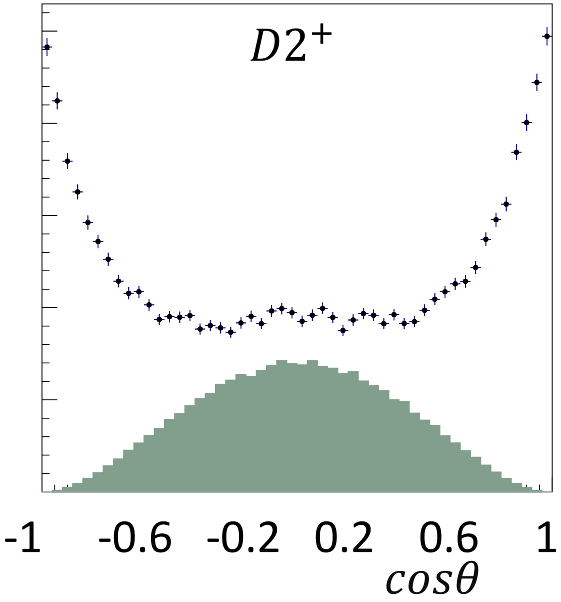






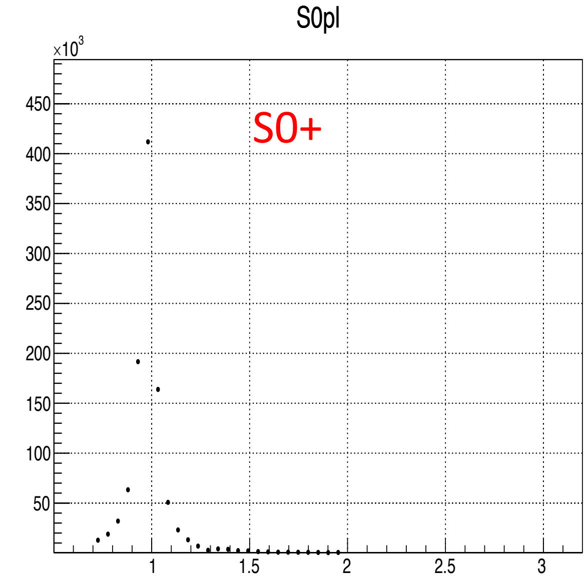


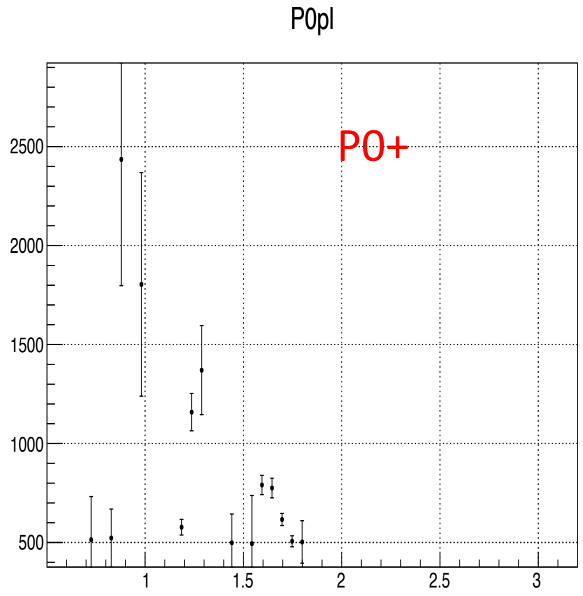


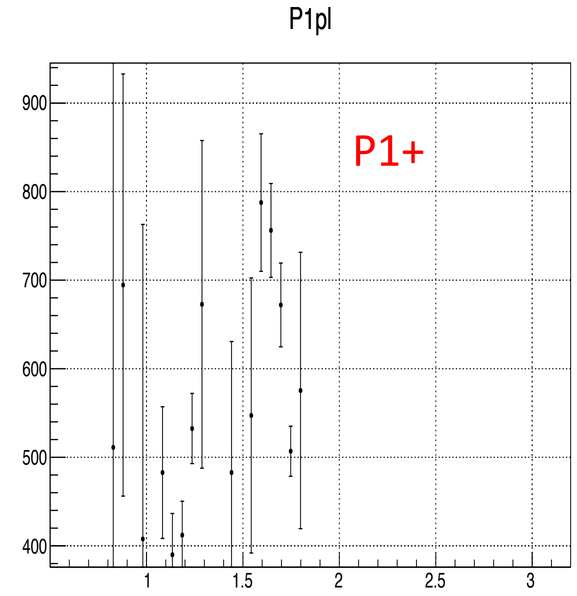


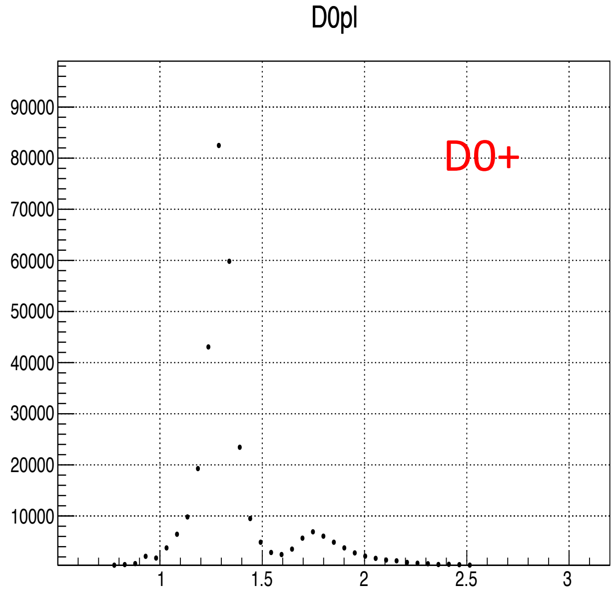
Fit 1 results (fitting in M and t bins)

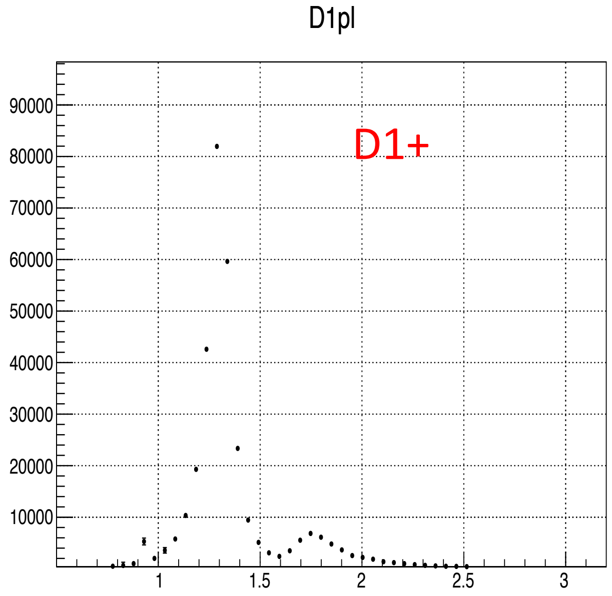
Amplitudes used in fitting are S0-, P0+, P1+, D0+, D1+, D2+. Good starting values for fit parameters

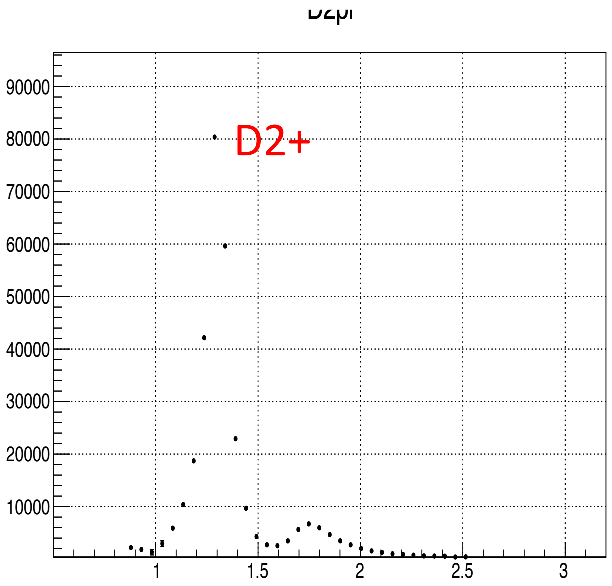


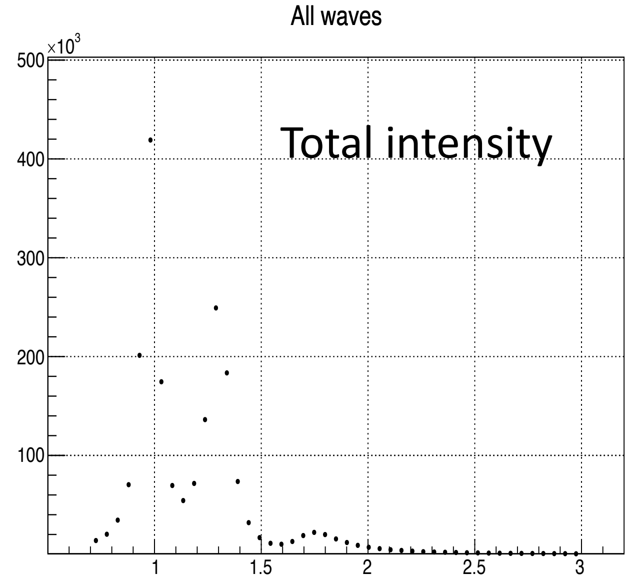












Analysis strategy

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$${}^{(\epsilon)}\rho^{1,\ell\ell'}_{mm'} = -\epsilon\kappa \sum_k \Big((-1)^m [\ell]^{(\epsilon)}_{-m;k} [\ell']^{(\epsilon)*}_{m';k} + (-1)^{m'} [\ell]^{(\epsilon)}_{m;k} [\ell']^{(\epsilon)*}_{-m';k} \Big) ,$$

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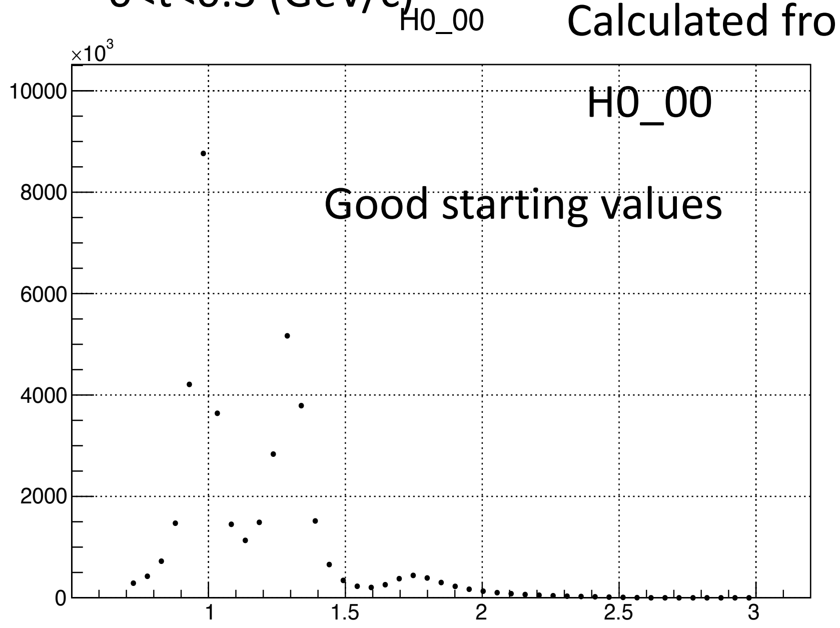
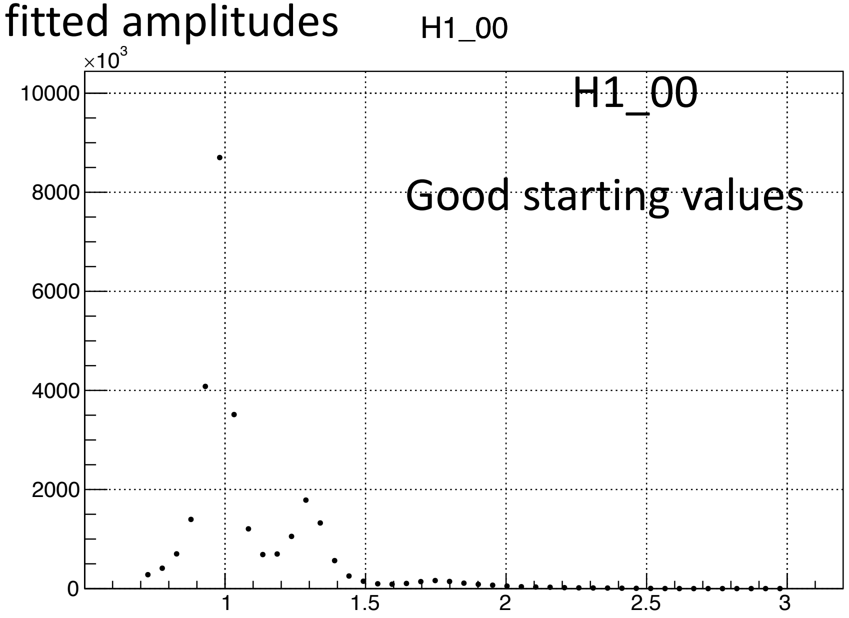
Polarized moments calculated with partial waves

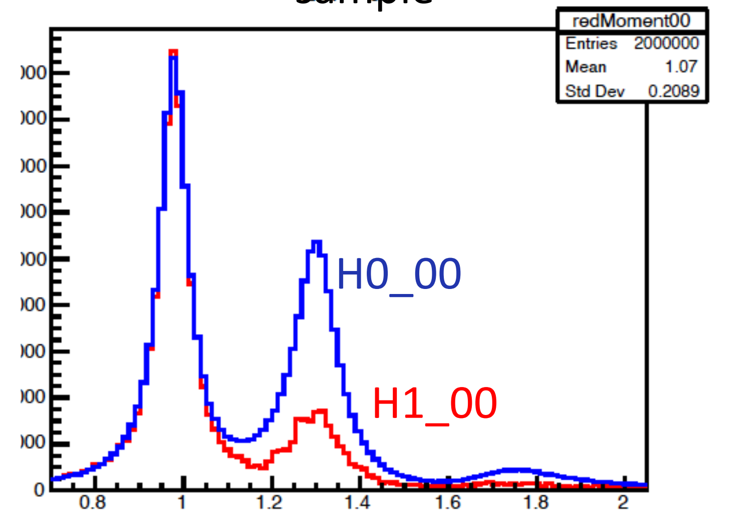
$0<t<0.3$ (GeV/c)^2

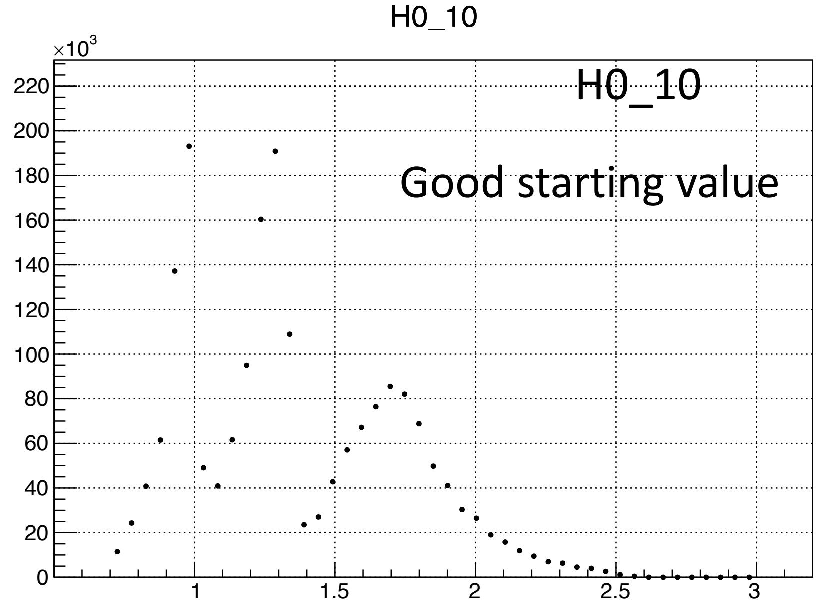
Calculated from fitted amplitudes

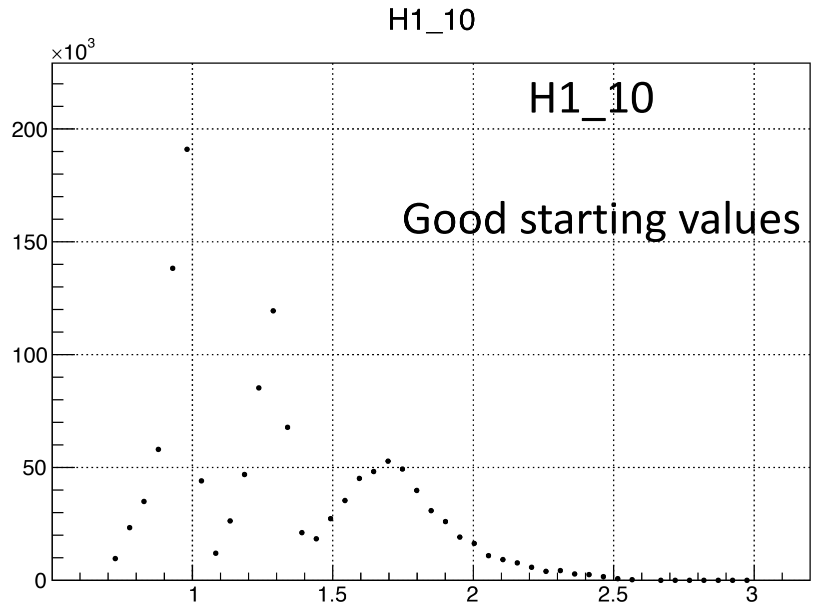
Obtained by weighting events of data sample

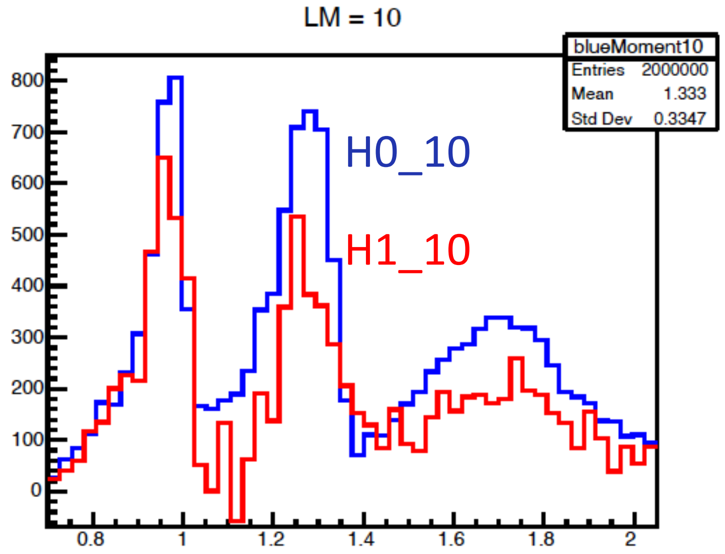












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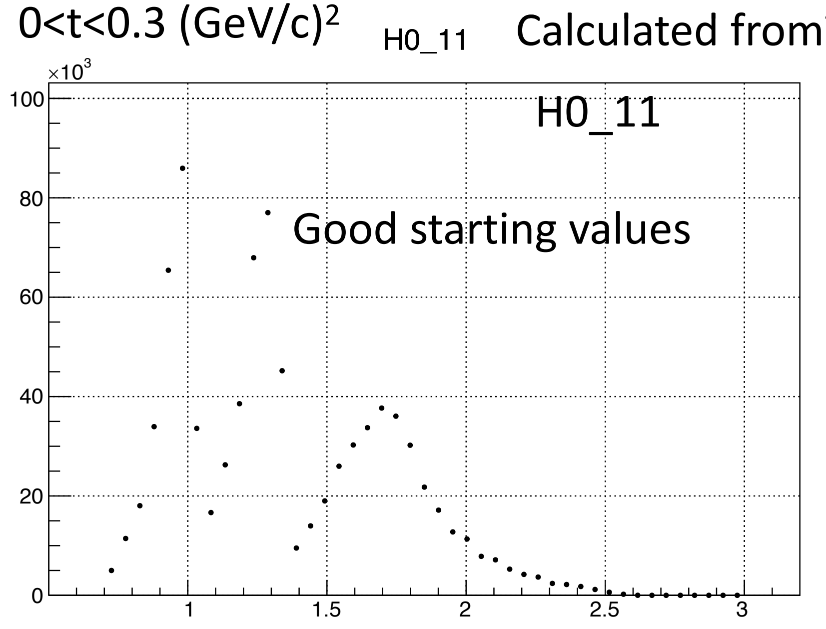

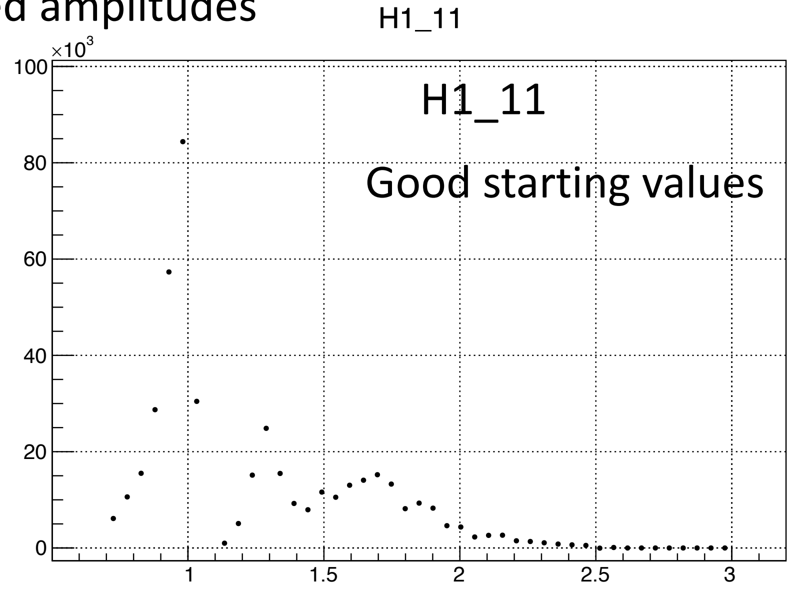

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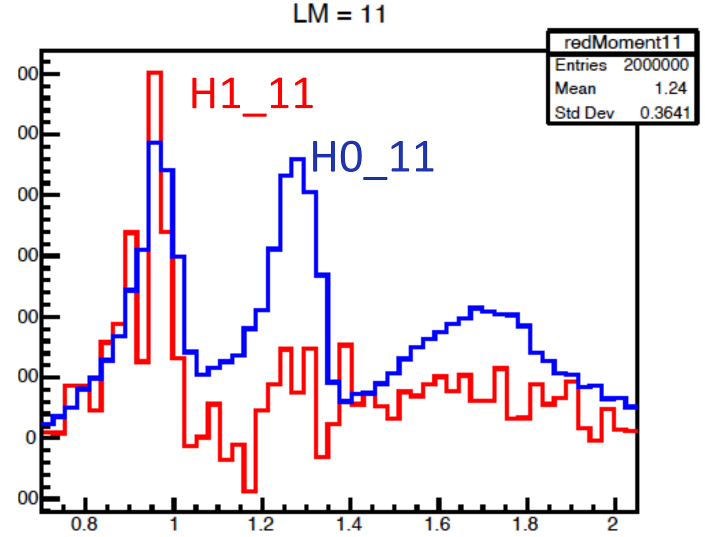

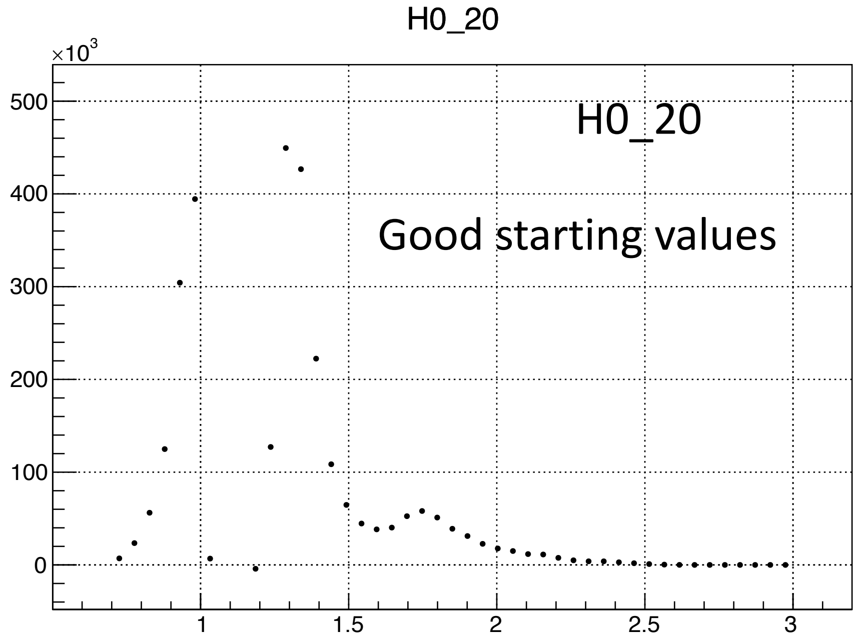

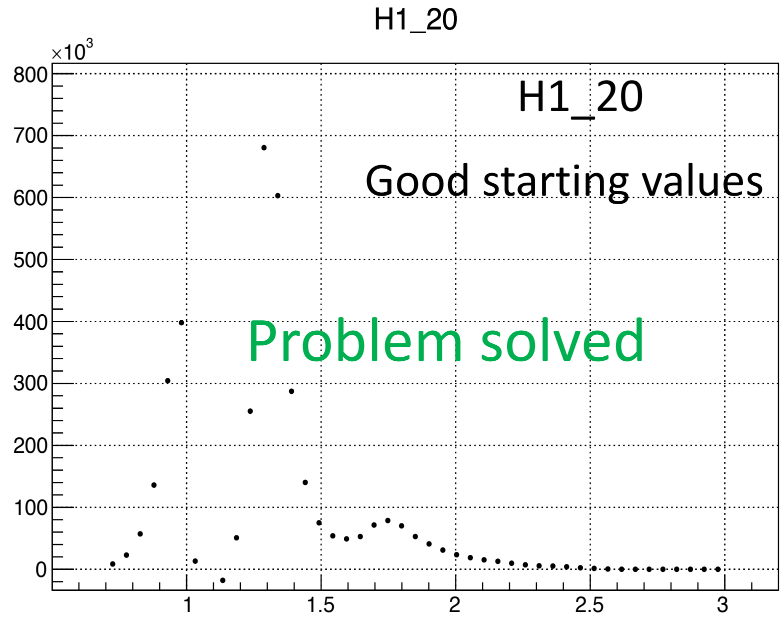

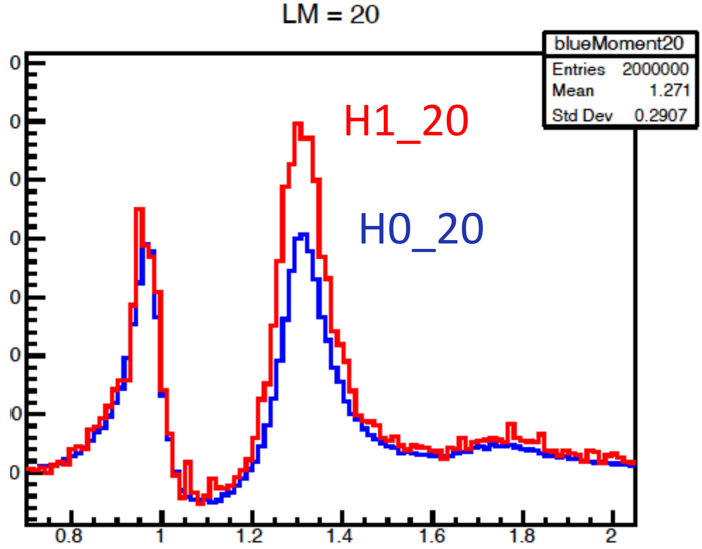

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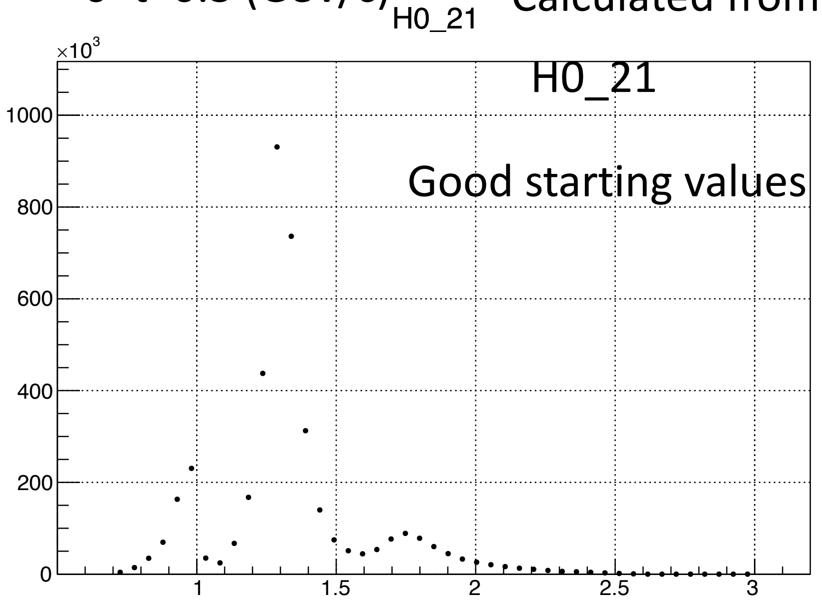

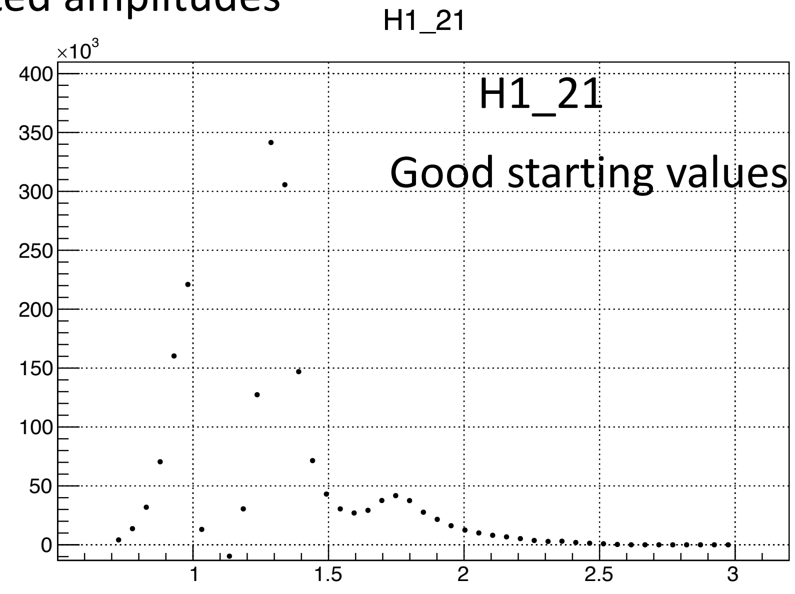

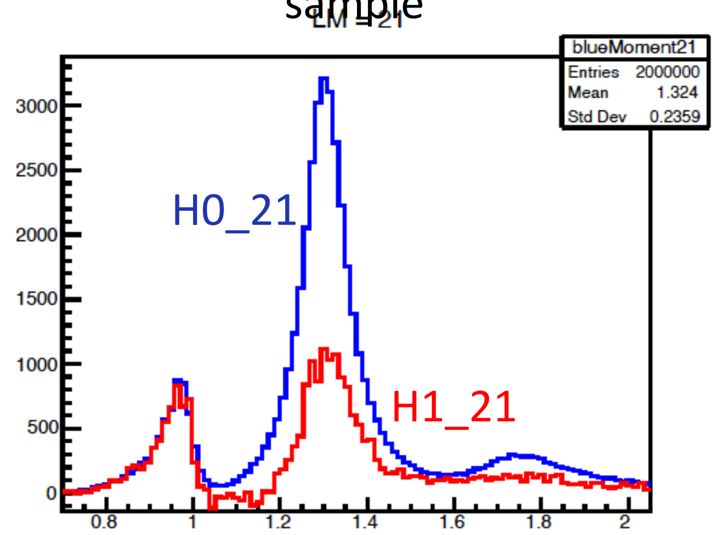

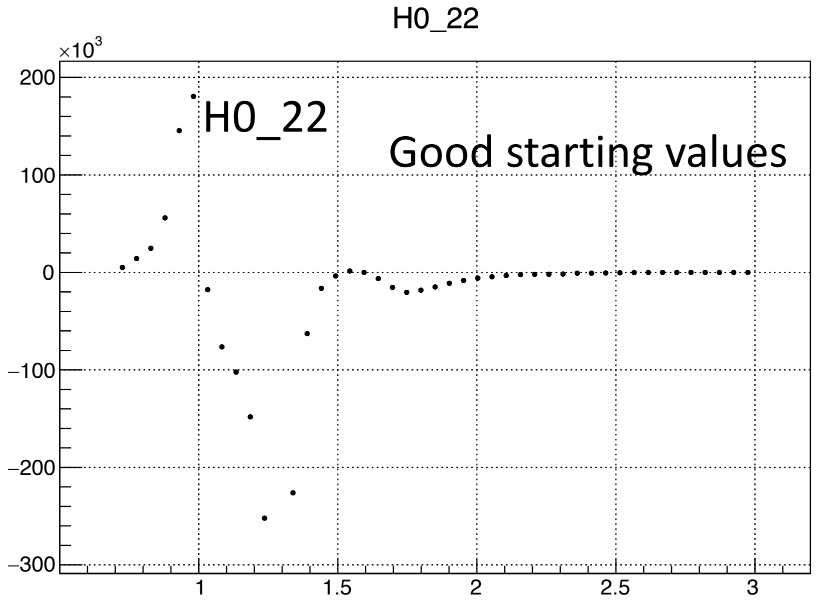

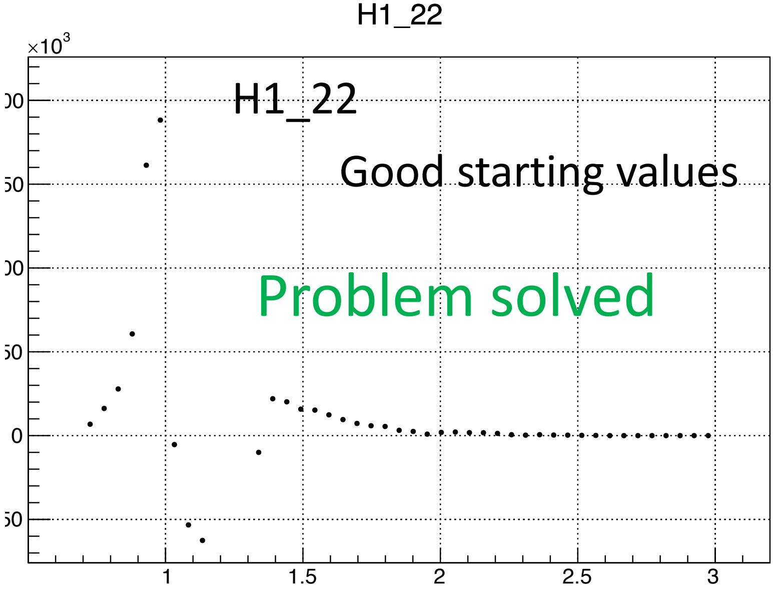

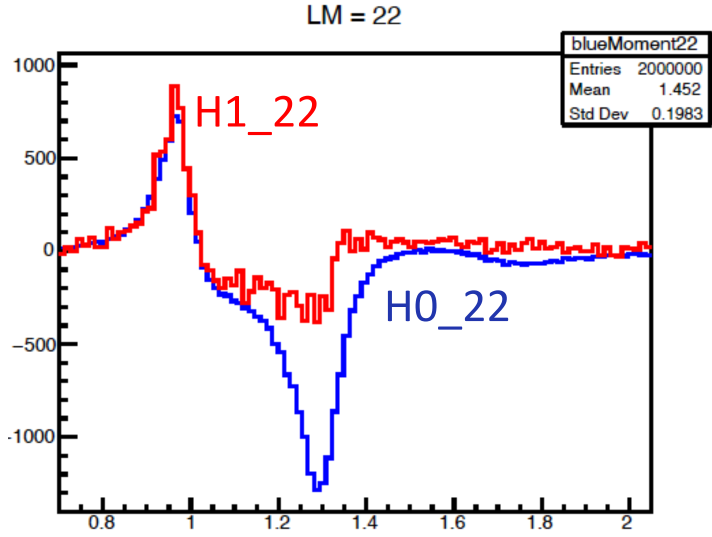

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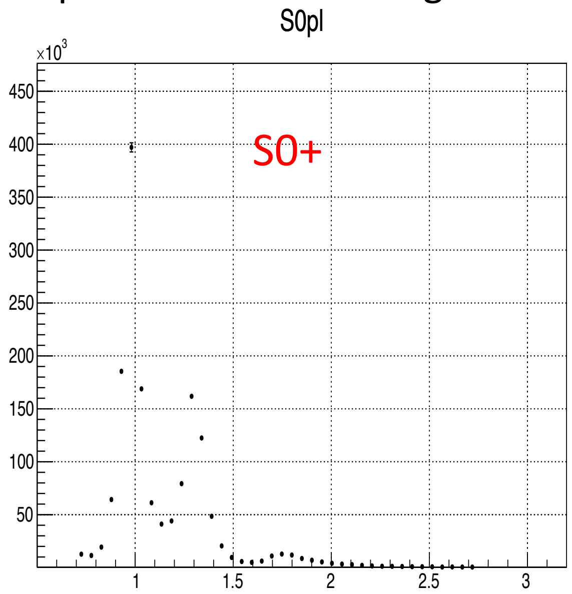
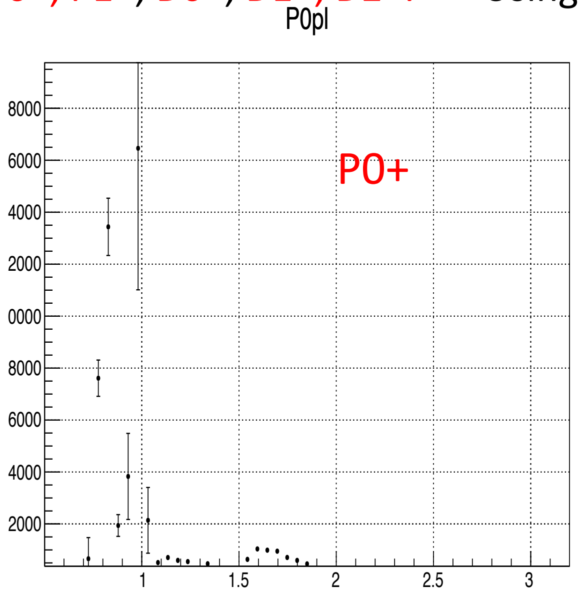
Fitting data with S0-, P0+, P1+, D0+, D1+, D2+ amplitude set

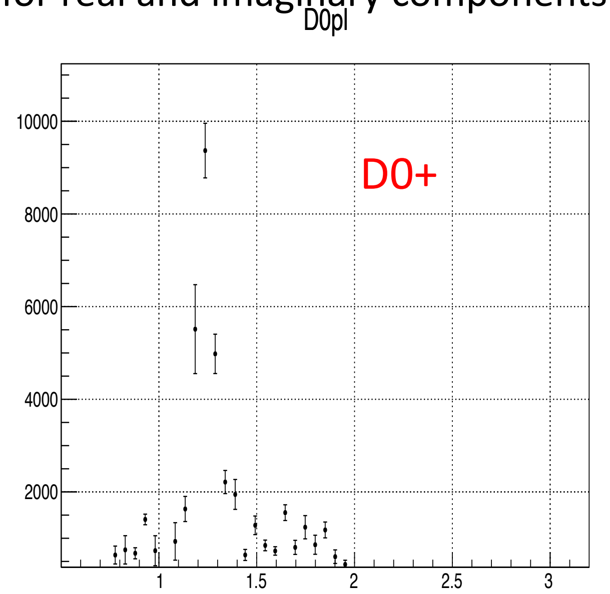
with S0-, P0+, D0+, D1+, G0+ ,G1+ .

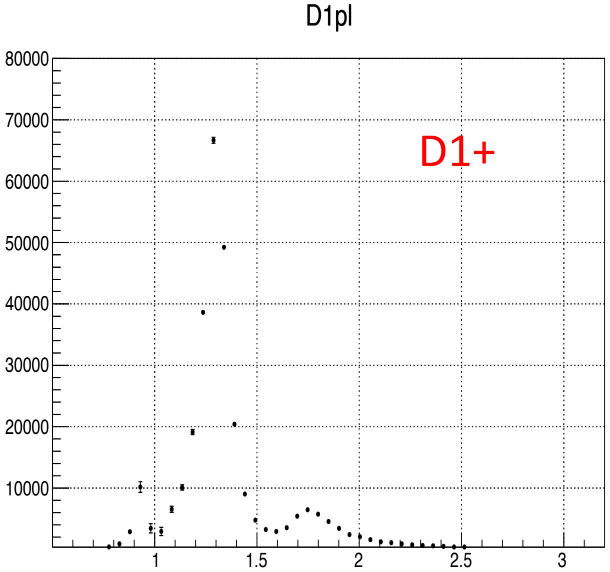
Fit 2 results (fitting in M and t bins)

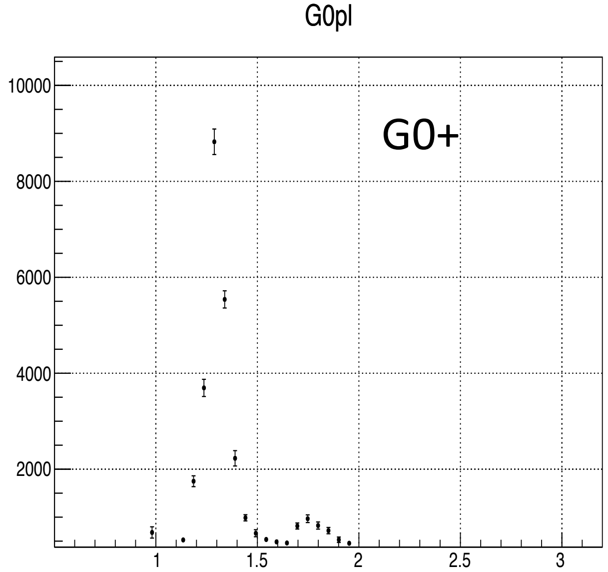
Amplitudes used in fitting are S0+, P0+, P1+, D0+, D1+, D2+.   Using 1 for real and imaginary components of the fit parameters

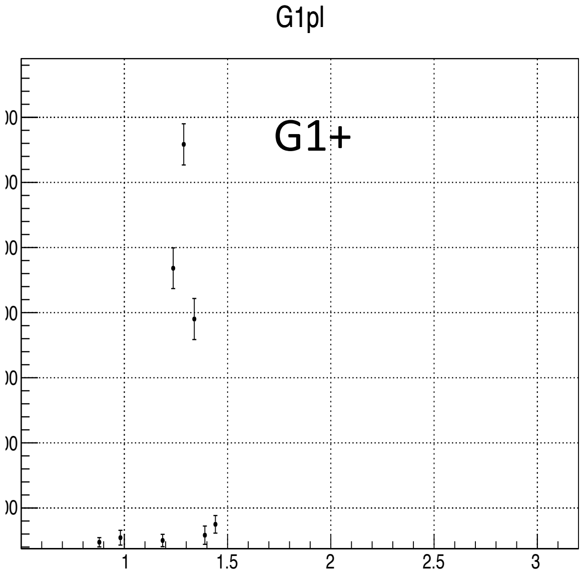


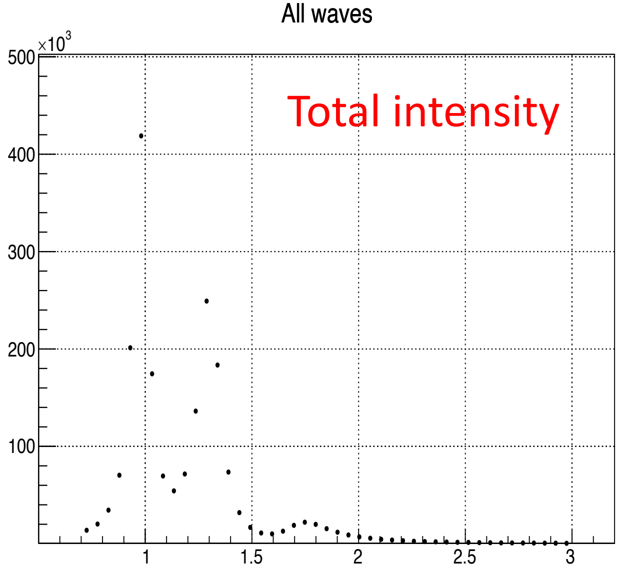




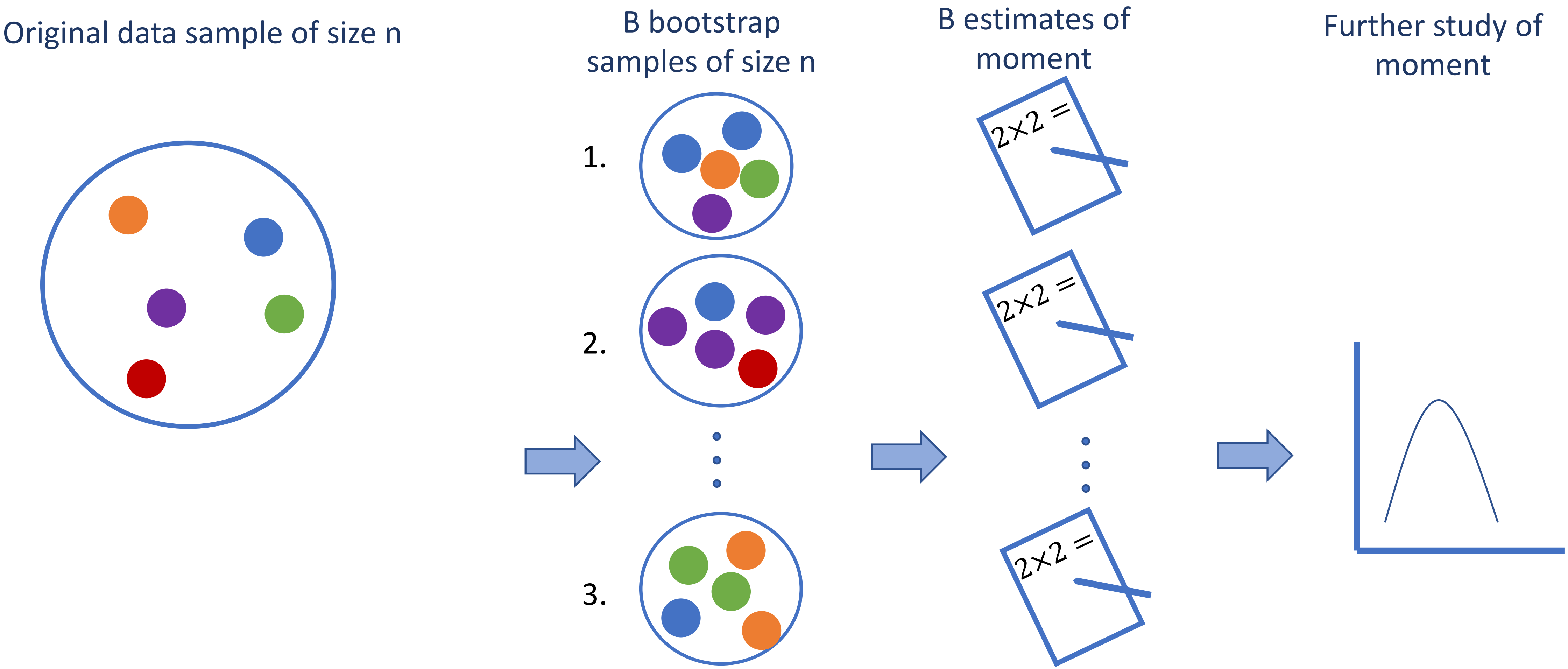








Bootstrapping method for estimation of uncertainties

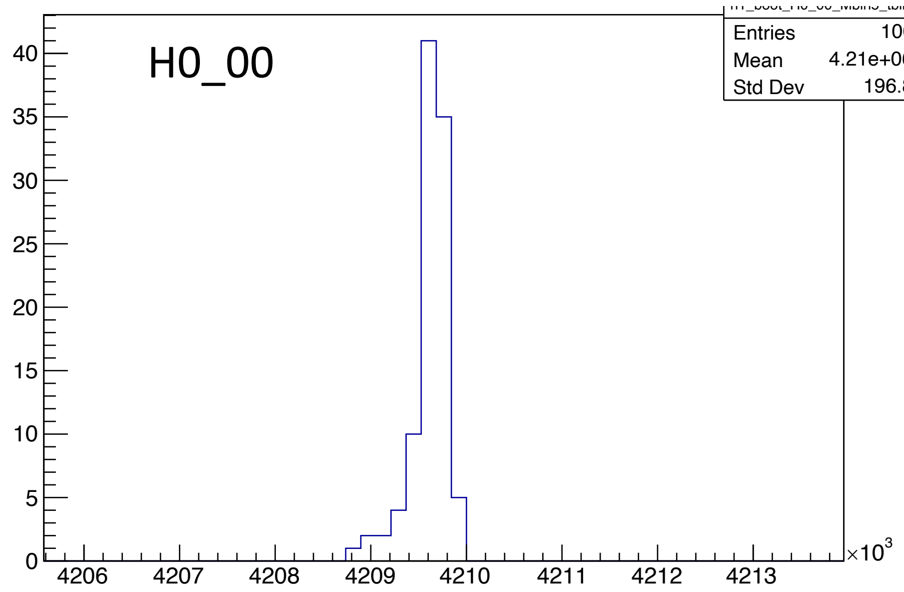


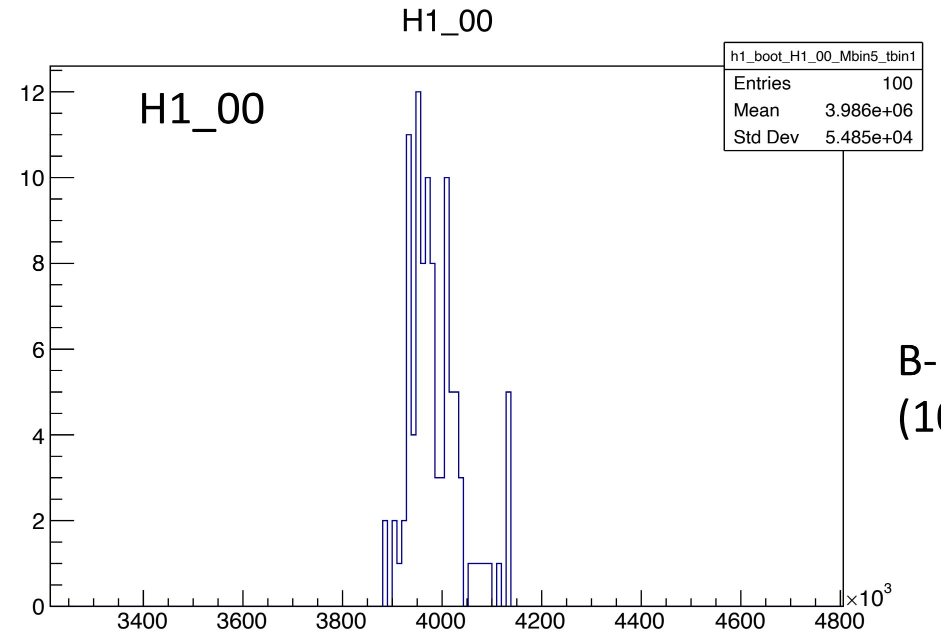
1. Draw a Bootstrap Sample from the original sample data with replacement with size n.
2. Evaluate intensity for each Bootstrap Sample which will result in B estimates of intensity.
3. Construct a histogram of B estimates of intensity and use it to make further statistical inference, such as:

    - Estimating the standard error of statistic for Intensity.

Distributions of moment values from 100 bootstrapping samples for **M bin=5 and t bin=1**

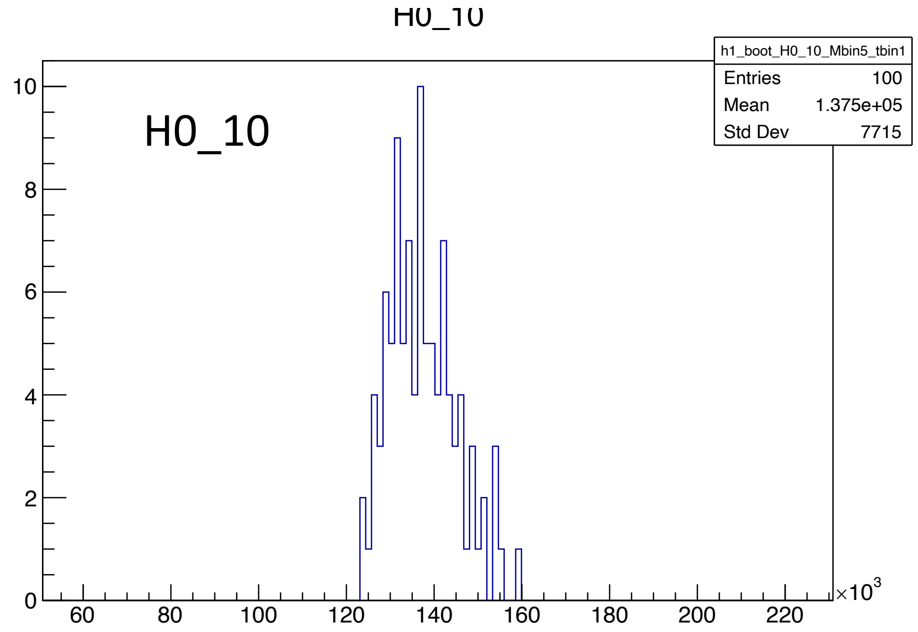
M~0.93 GeV/c^2 0<t<0.3 (GeV/c)^2

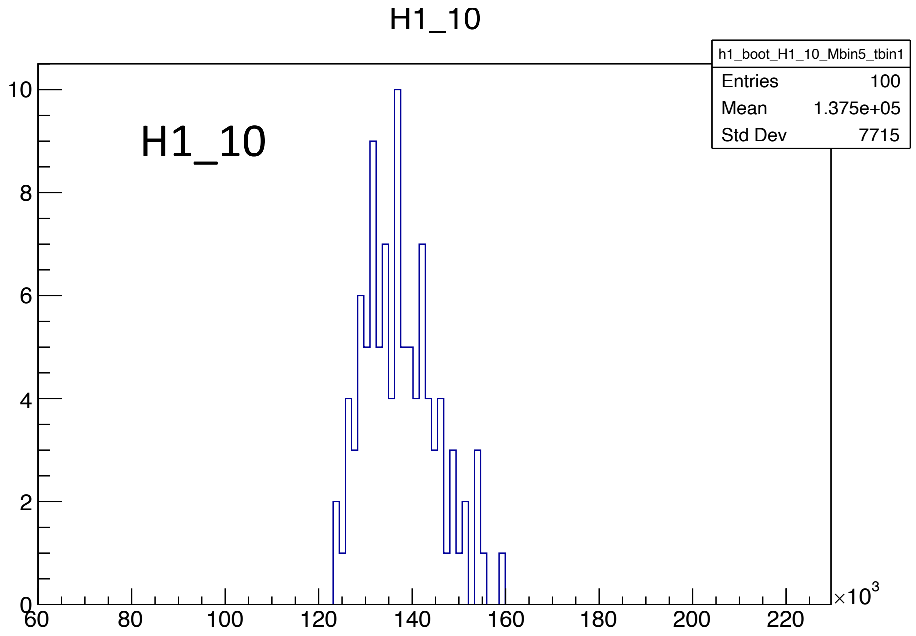




$$\sigma = \sqrt{\frac{\sum_i^B (I_i - I_{mean})^2}{B}}$$

B- number of Bootstraps (100 in this case)



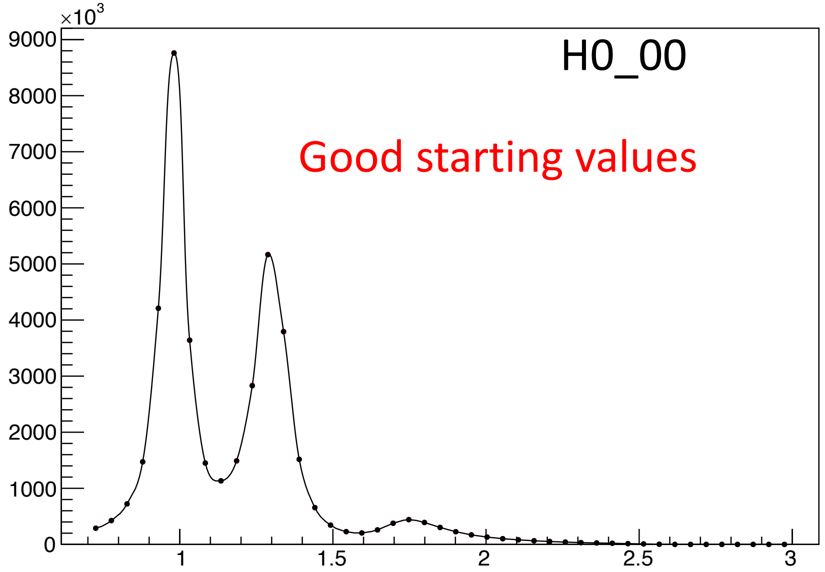


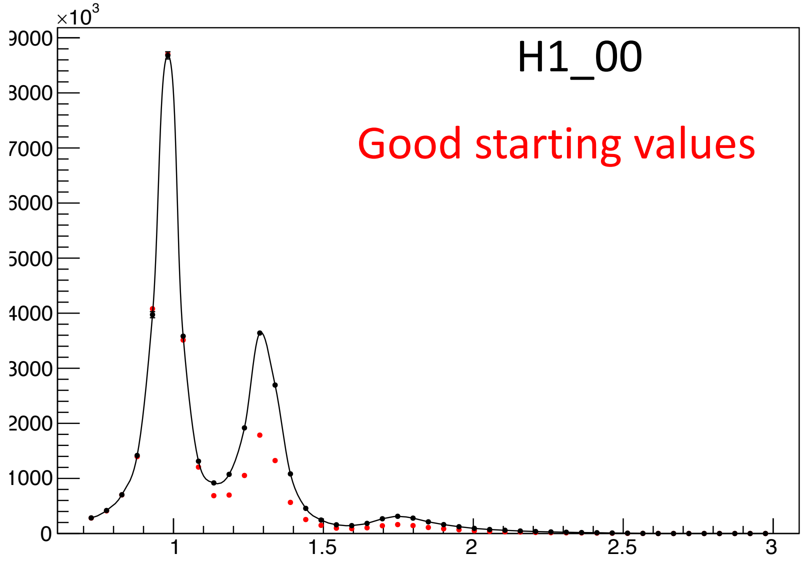
Polarized moments calculated with partial waves with uncertainties from bootstrapping

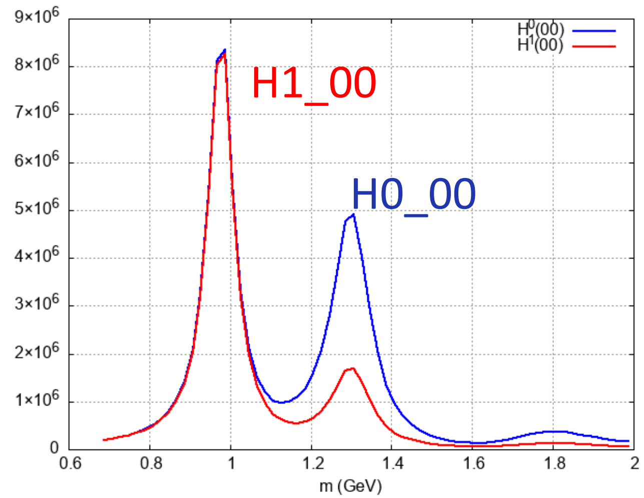
0<t<0.3 (GeV/c)^2

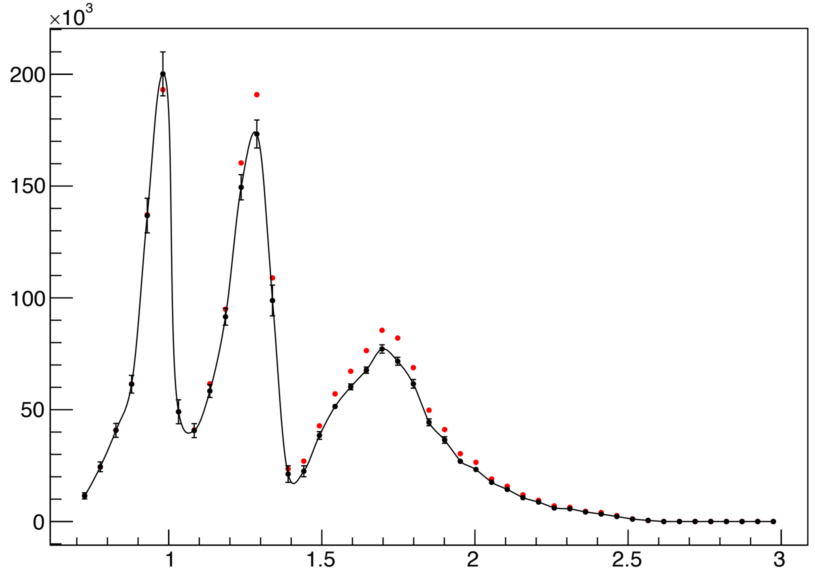
Calculated from fitted amplitudes, with bootstrapping uncert.

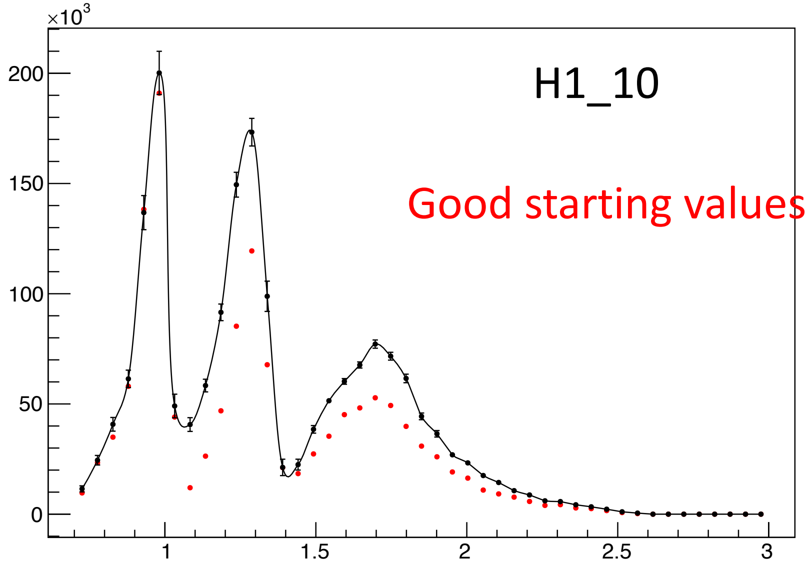
Obtained from Vincents codes

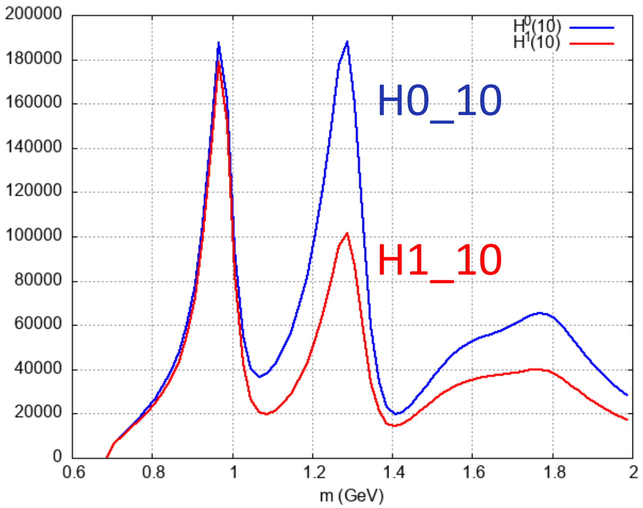








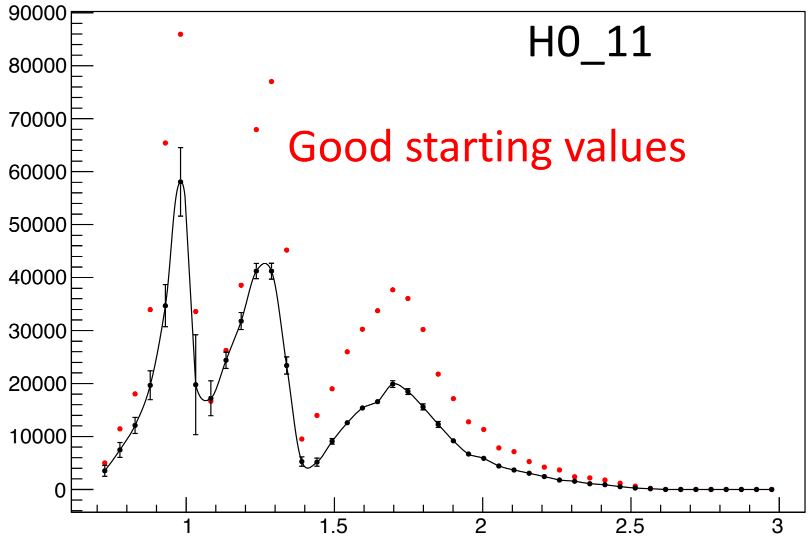


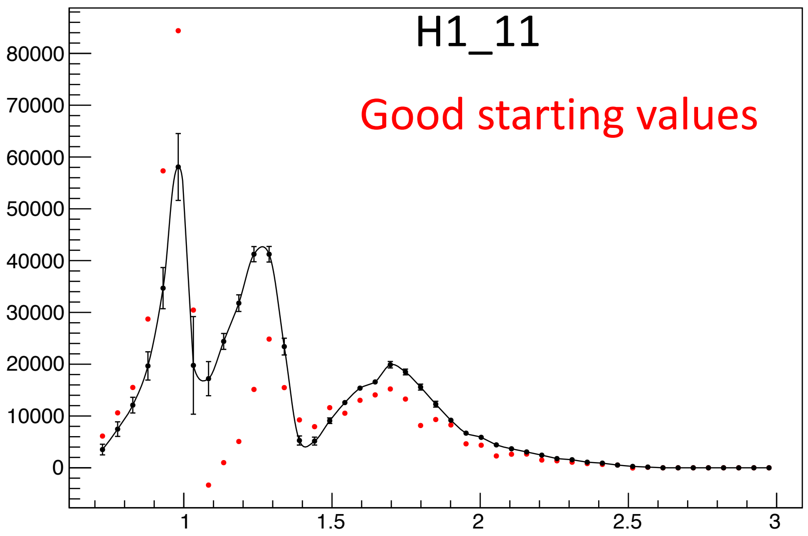


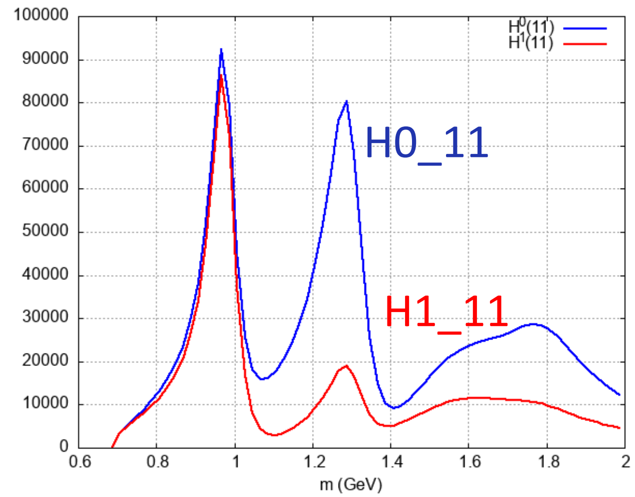
Polarized moments calculated with partial waves with uncertainties from bootstrapping

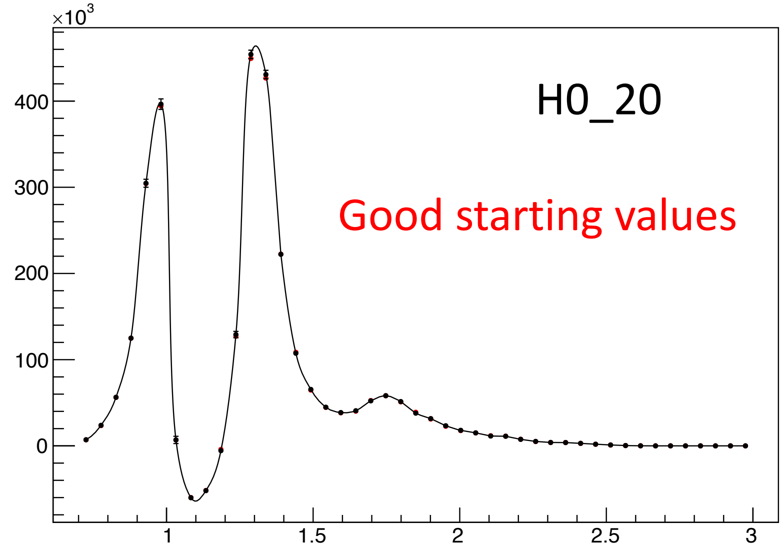
$0<t<0.3$ (GeV/c)^2 Calculated from fitted amplitudes , with bootstrapping uncert.

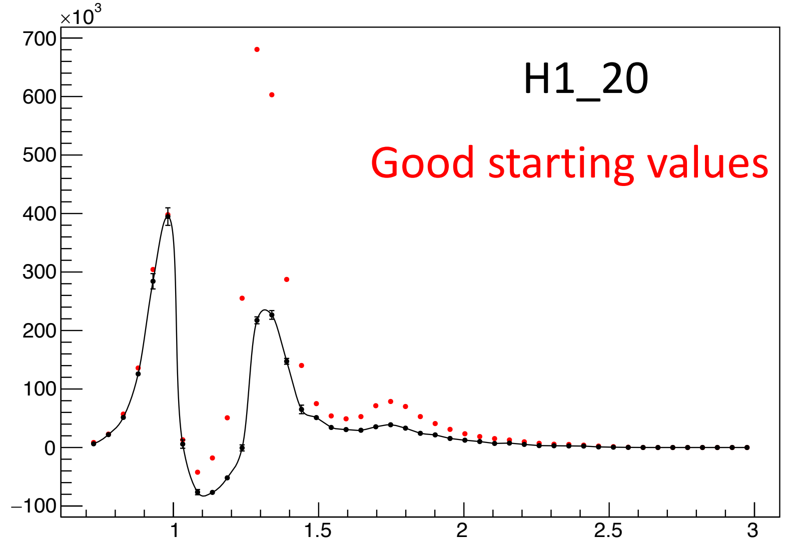
Obtained by weighting events of data sample

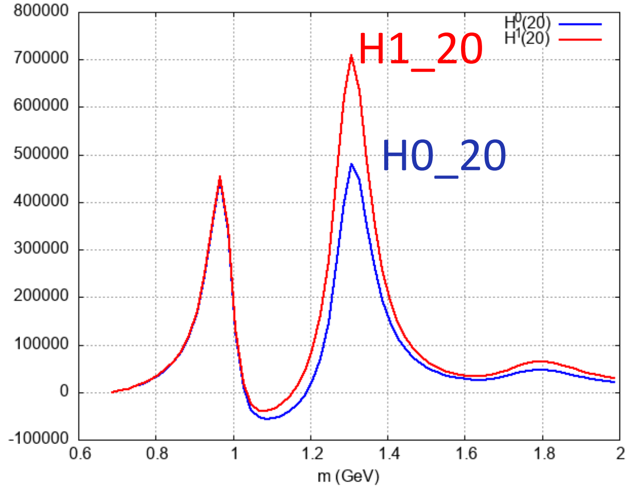












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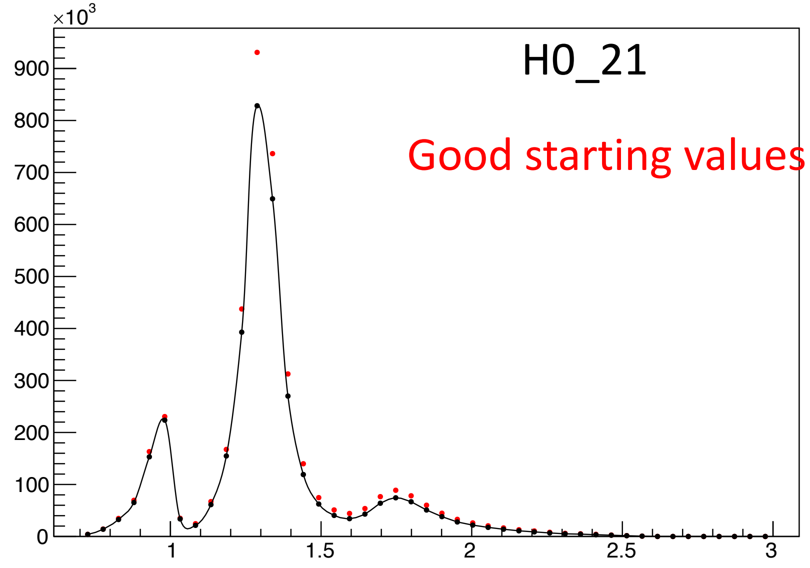
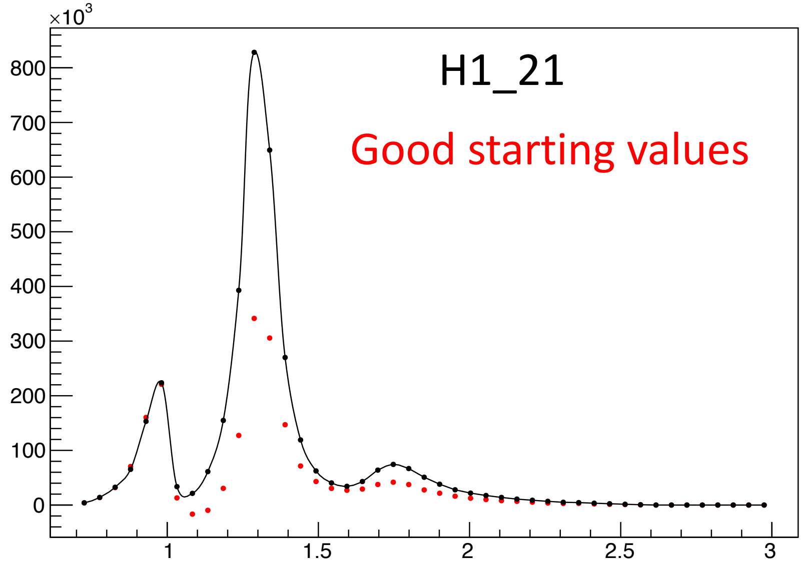
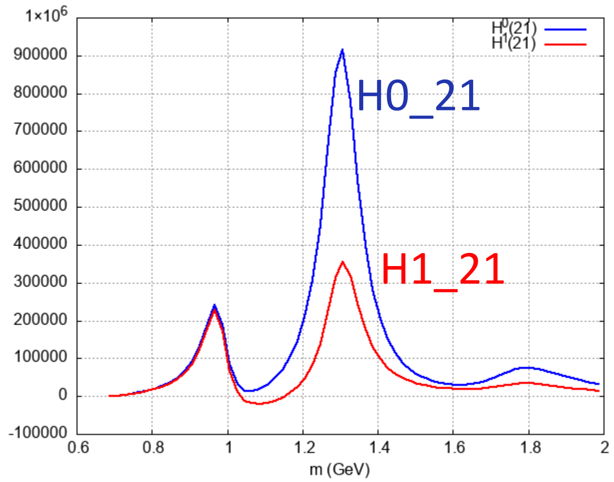
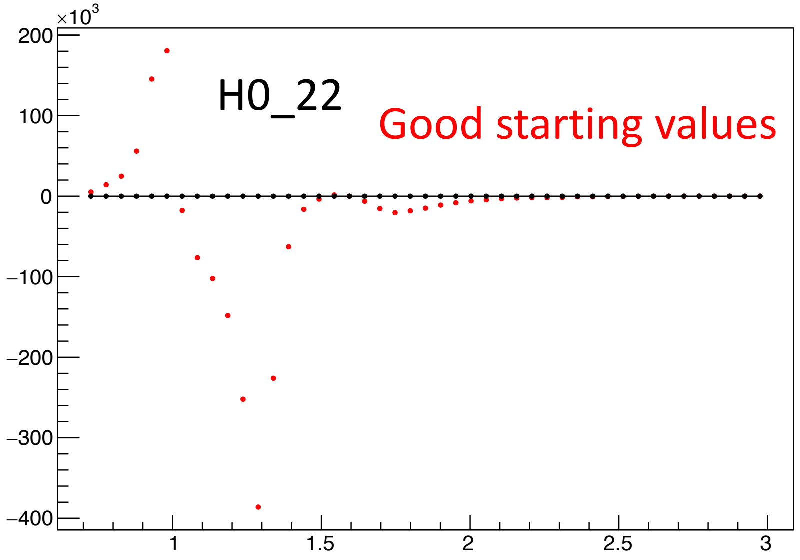
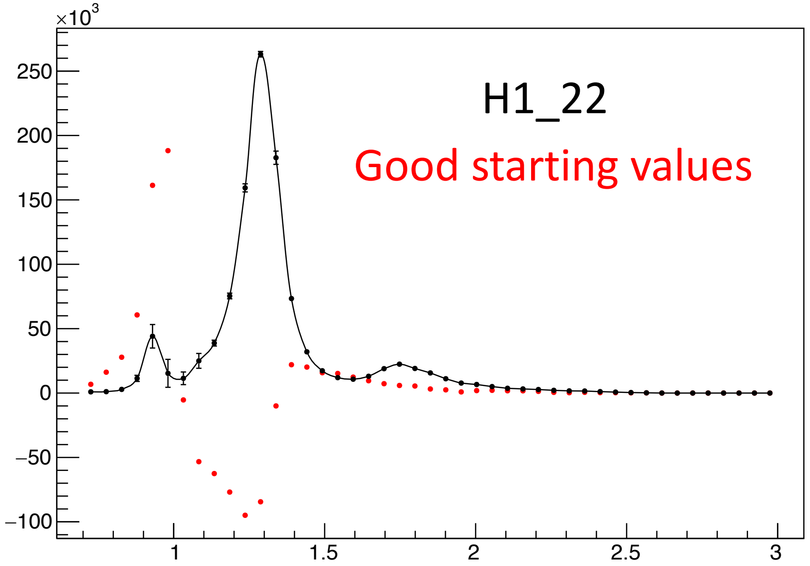
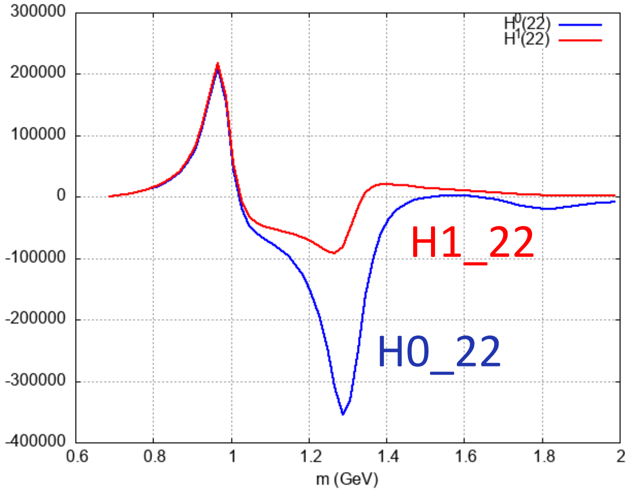










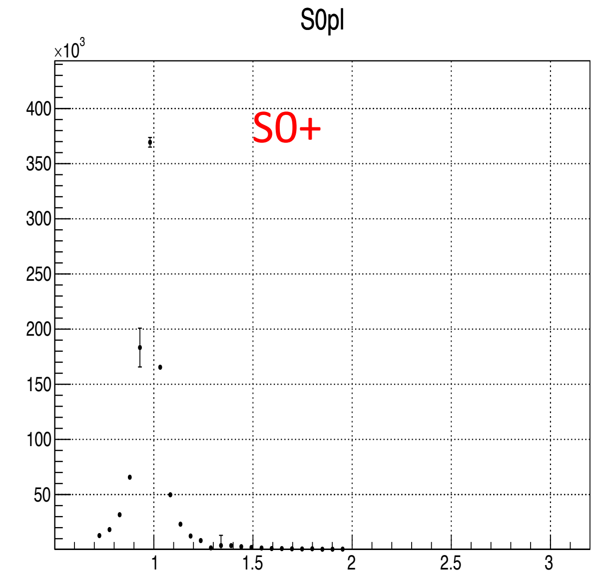


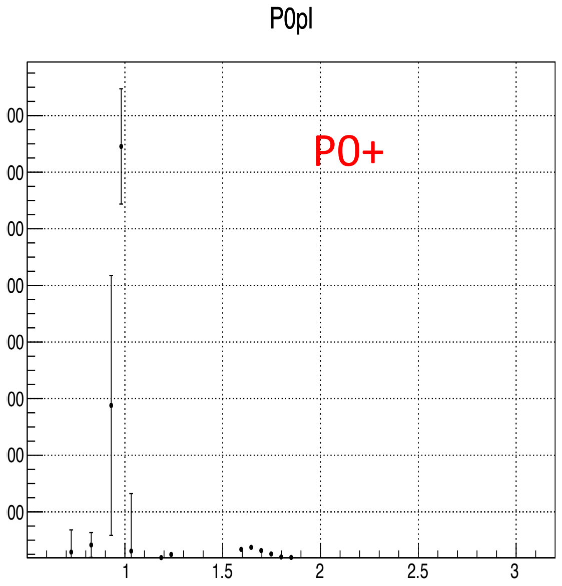
Fitting data with S0-, P0+, P1+, D0+, D1+, D2+ amplitude set

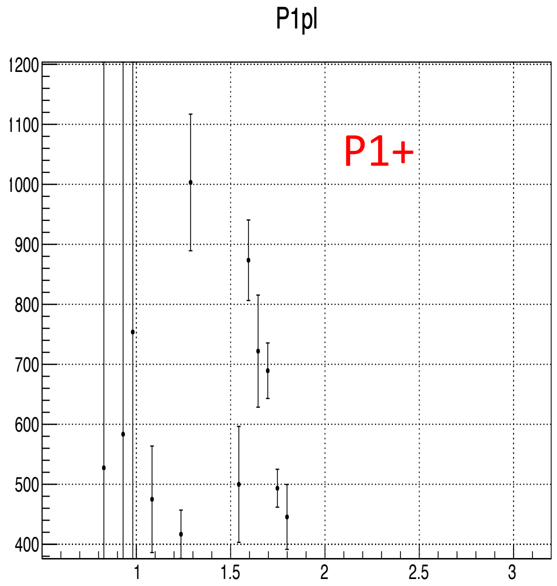
with S0-, P0+, P1+, D0+, D1+, D2+, G0+ ,G1+ .

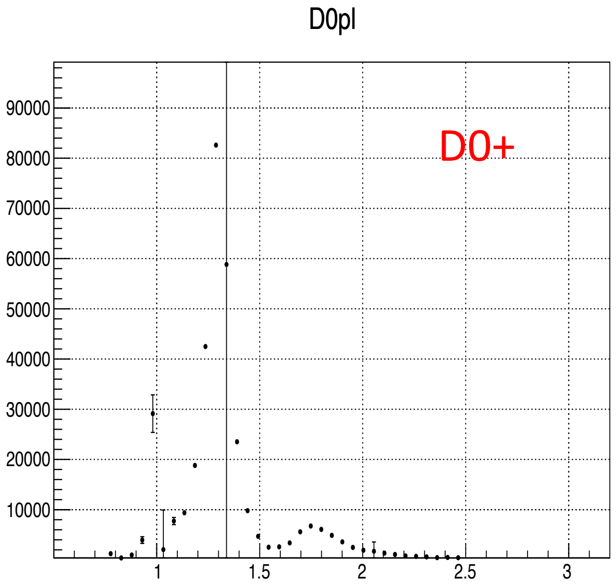
Fit 1 results (fitting in M and t bins)

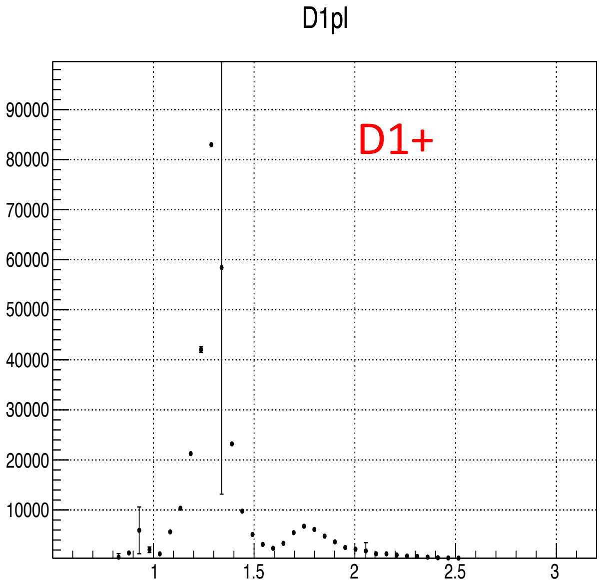
Amplitudes used in fitting are S0+, P0+, P1+, D0+, D1+, D2+. Good starting values for fit parameters

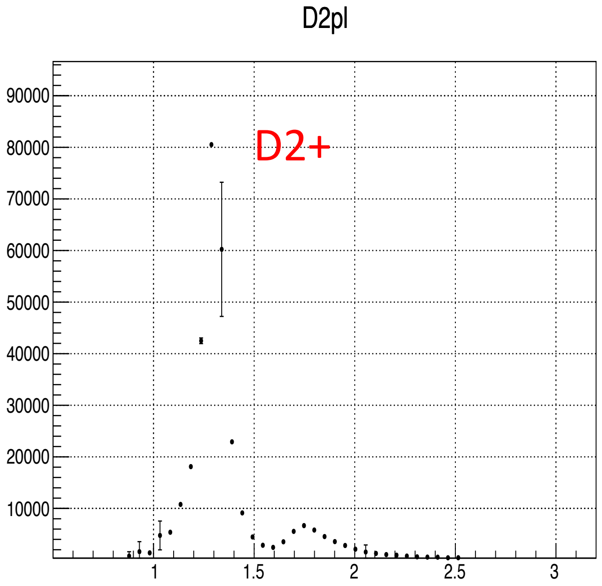


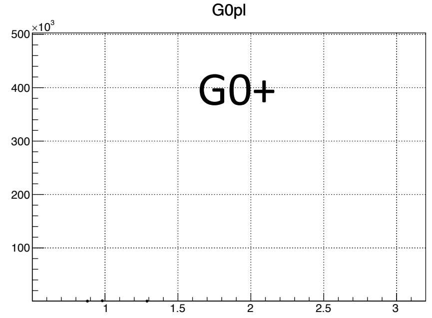


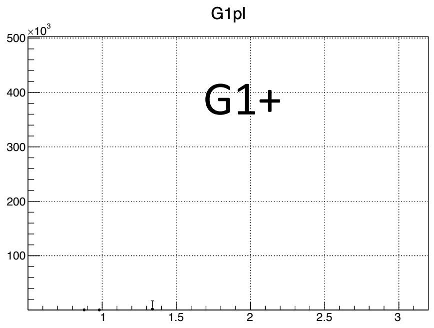


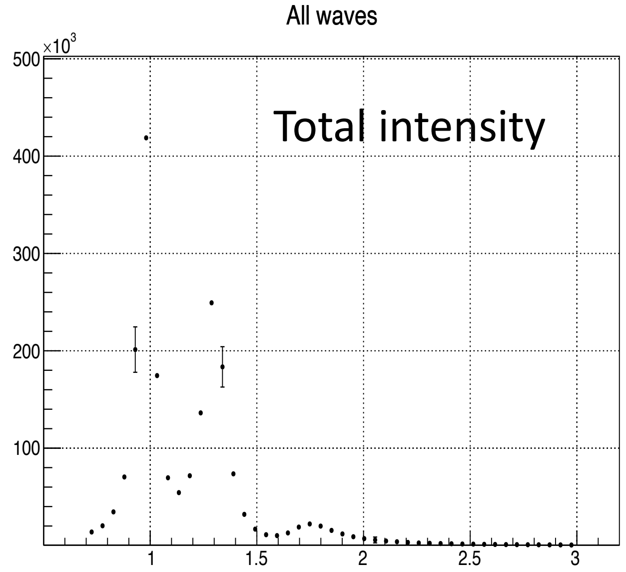






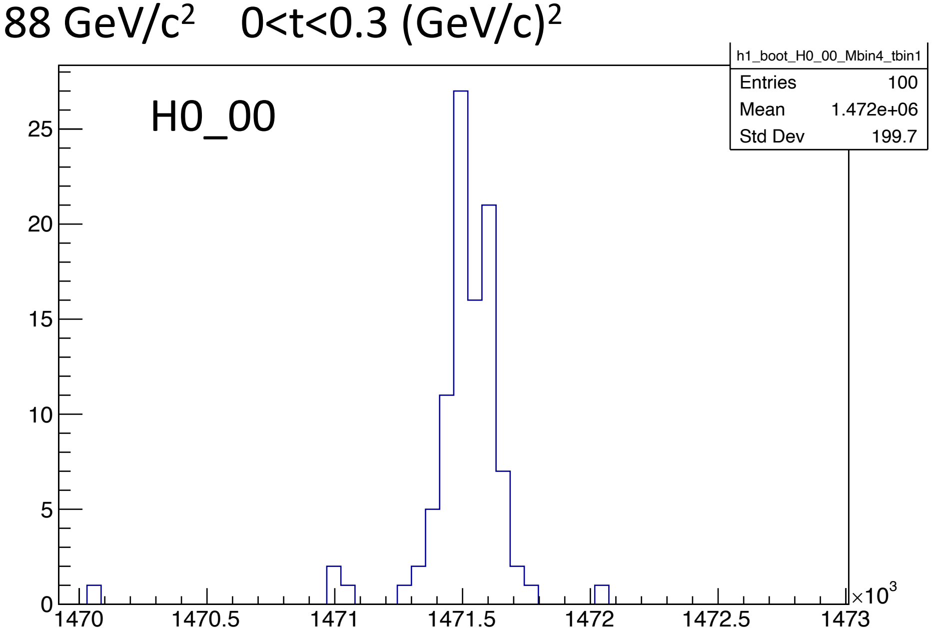


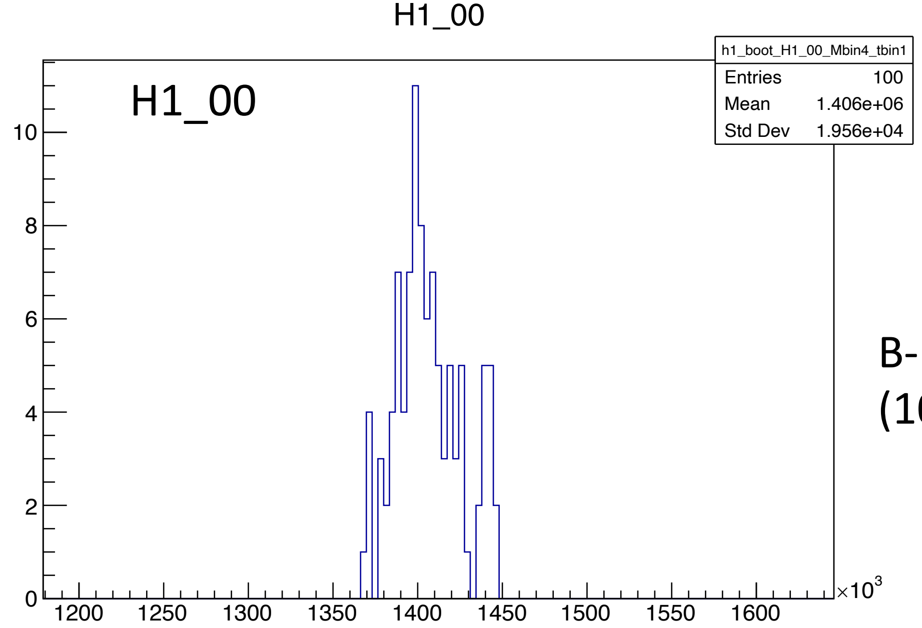




Distributions of moment values from 100 bootstrapping samples for **M bin=4 and t bin=1**

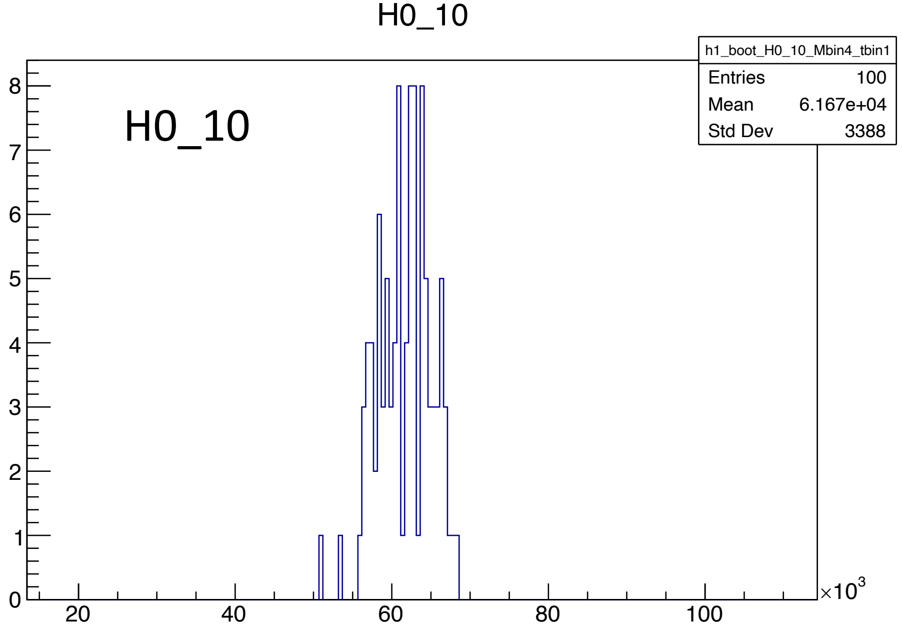
M~0.88 GeV/c^2 0<t<0.3 $(GeV/c)^2$

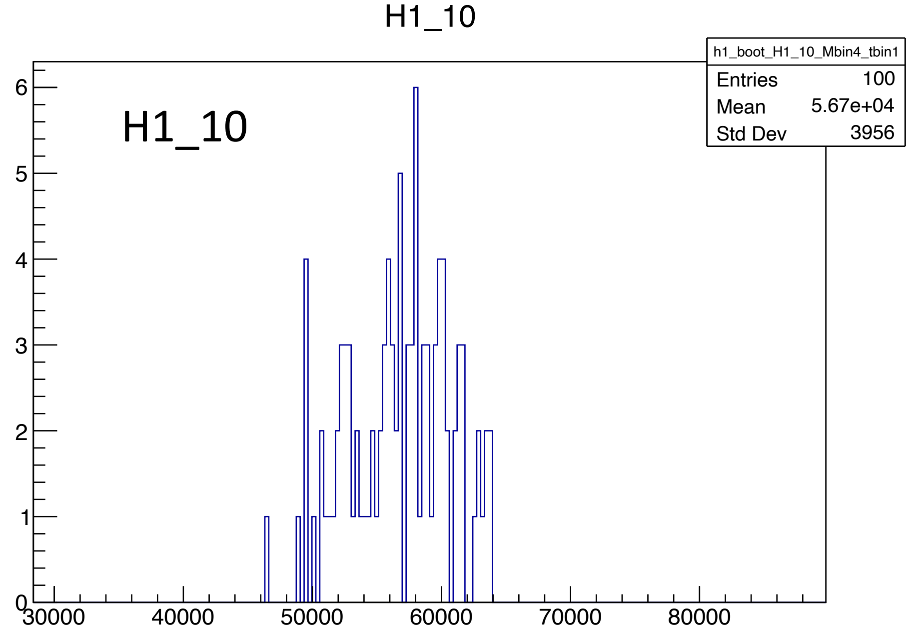




$$\sigma = \sqrt{\frac{\sum_i^B (I_i - I_{mean})^2}{B}}$$

B- number of Bootstraps (100 in this case)



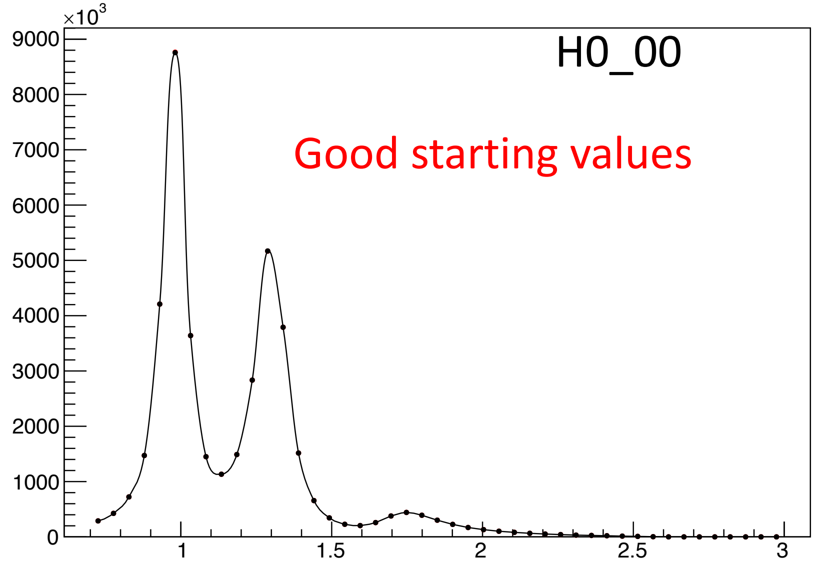


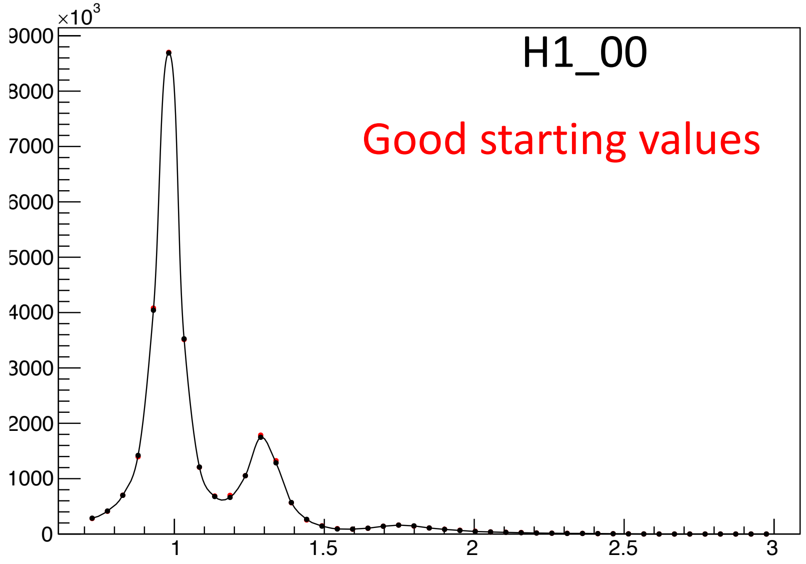
Polarized moments calculated with partial waves with uncertainties from bootstrapping

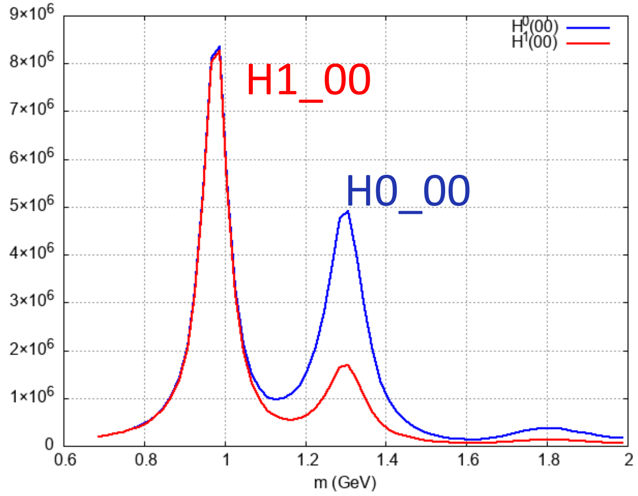
0<t<0.3 (GeV/c)^2

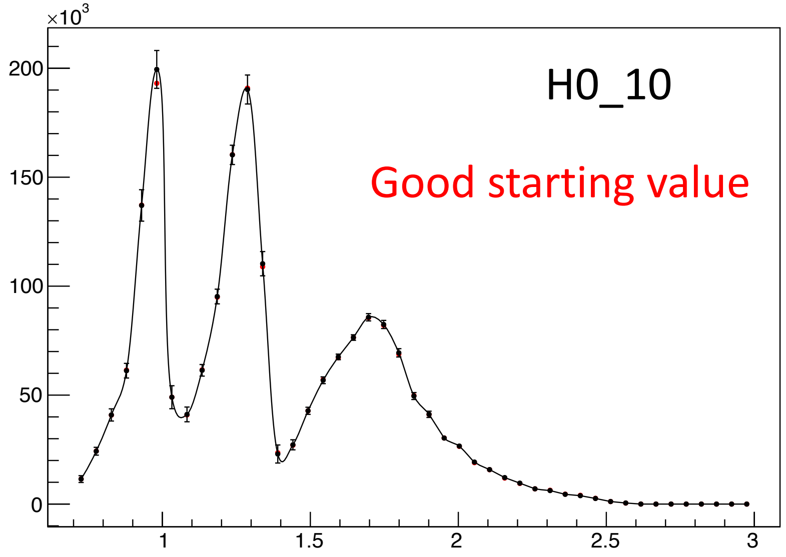
Calculated from fitted amplitudes, with bootstrapping uncert.

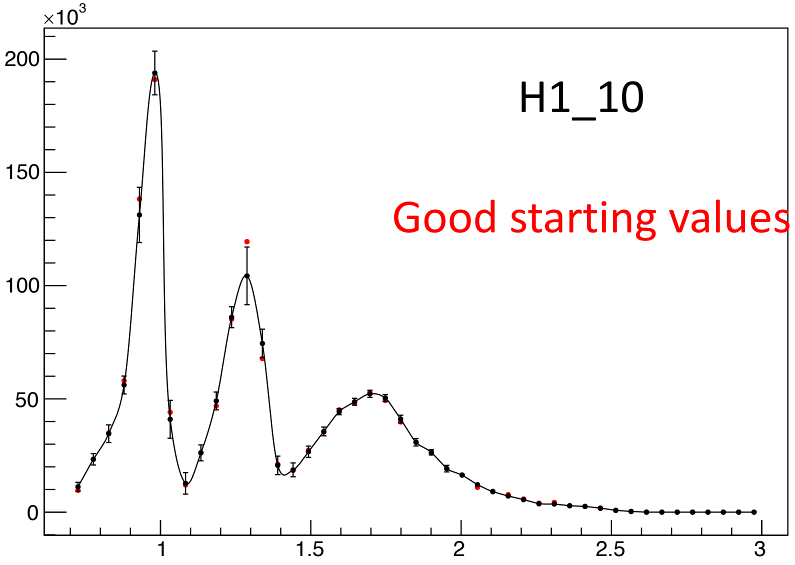
Obtained from Vincents codes

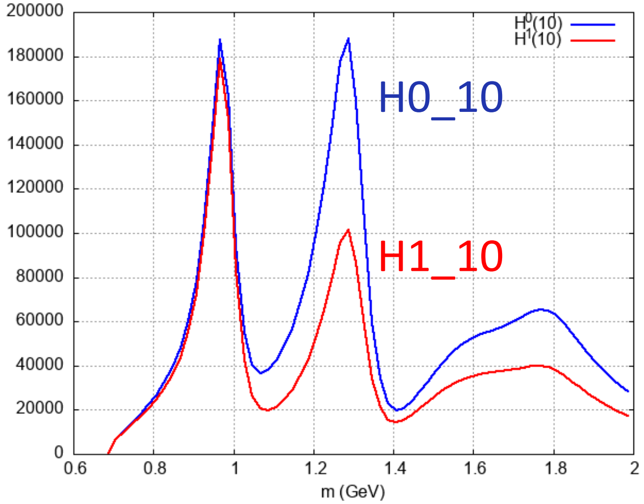








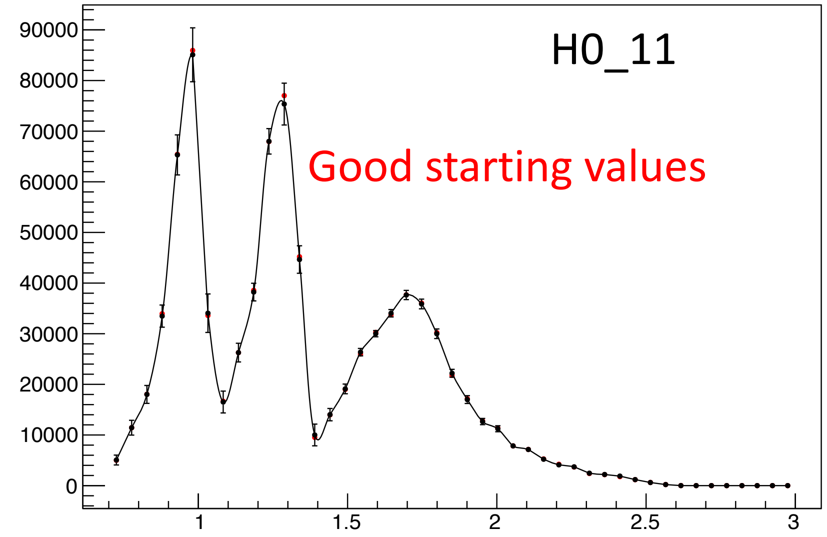


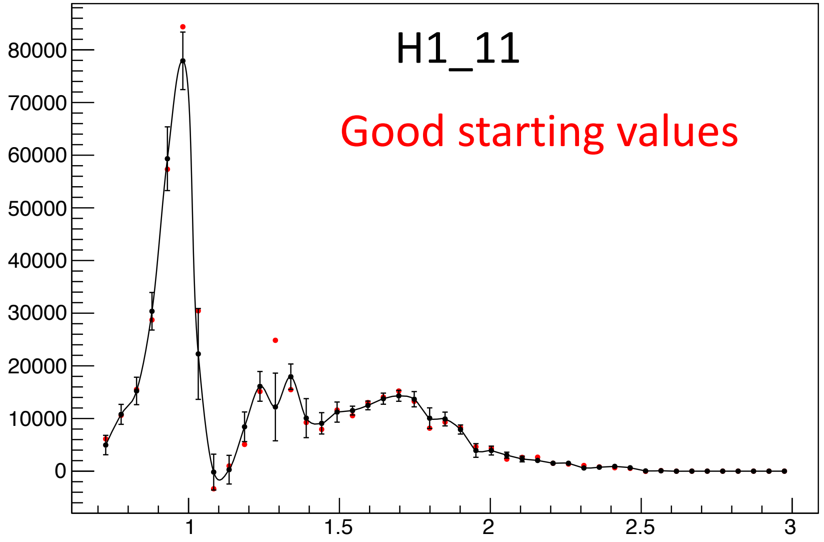


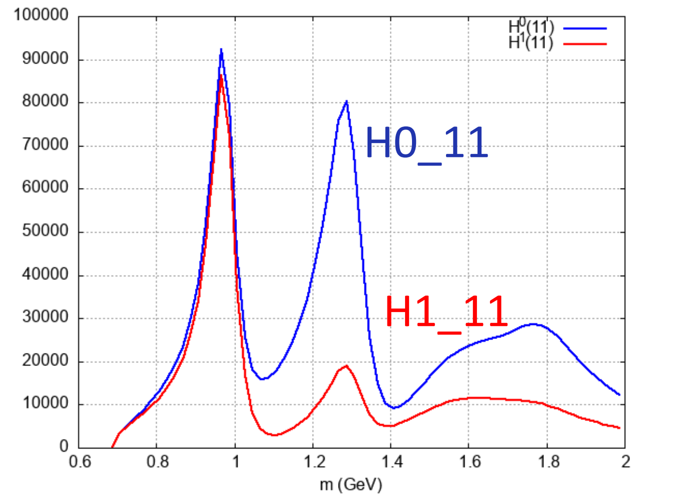
Polarized moments calculated with partial waves with uncertainties from bootstrapping

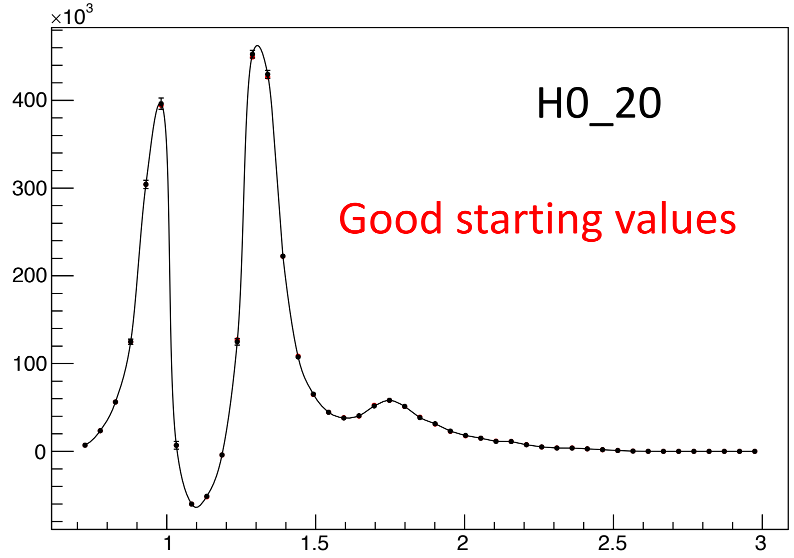
$0<t<0.3$ (GeV/c)^2 Calculated from fitted amplitudes , with bootstrapping uncert.

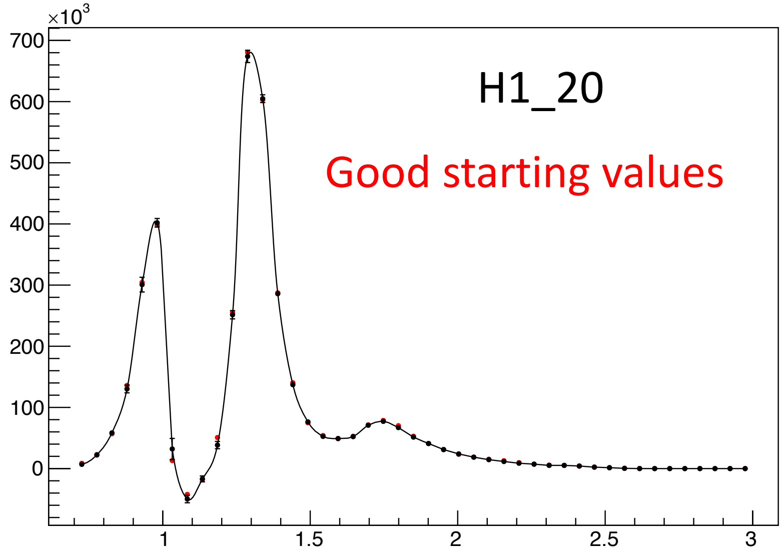
Obtained by weighting events of data sample

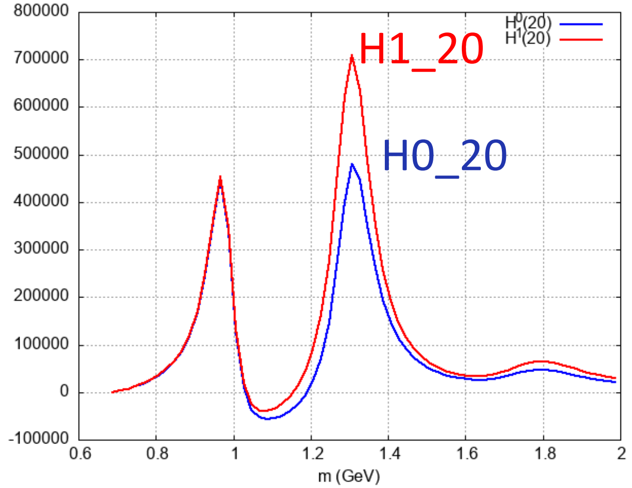












Polarized moments calculated with partial waves with uncertainties from bootstrapping

$0<t<0.3$ (GeV/c)^2 Calculated rom fitted amplitudes , with bootstrapping uncert.

Obtained by weighting events of data sample