The Kalman Filter

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Motivation

- Understand the origin of χ^2 of each track
- Need to know the inner workings of the Kalman filter
- essential to physics analyses

Goal

What is the Kalman filter, and how does it work?

Overview I

- Kalman filtering is a powerful tool to estimate the best parameters for a given system
- The system is assumed to evolve according to a known set of equations, with some stochastic error
- The parameters to determine are x (n-dimensional vector)
- If the x are known at the $k-1$th point, then the values at the kth point are given as

$$x_k = Ax_{k-1} + w_{k-1},$$

where the matrix A represents the known dynamics, and the w_{k-1} represent the update error

Overview II

- At the kth point, we also do a measurement of the system, so that we have the prediction and the measurement
- In general, we cannot directly measure the observables of interest x directly, but we measure z (m-dimensional vector), which have the relation

$$z_k = Hx_k + v_k,$$

where H is a $m \times n$ matrix

- The v_k are the measurement errors associated with the measurement

The question is:

Given the prediction for x_k and the measurement z_k, what is the best estimate for x_k?

Simple Example I

- Assume we want to estimate the position of a particle in 1D
- Assume no dynamics
- If the 1st measurement yields $x_1 \pm \sigma_1$, our best estimate at a later time will be the same
- Now, if a second measurement yields $x_2 \pm \sigma_2$, what is the best estimate for x?

$$\Downarrow$$

Best estimate is *weighted average*

$$x = \frac{\sum_{i=1,2} x_i/\sigma_i^2}{\sum_{i=1,2} 1/\sigma_i^2}$$

$$\sigma^2 = 1/\left(\sum_{i=1,2} 1/\sigma_i^2\right)$$

Simple Example I

- The answer is:

$$x = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} x_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} x_2$$
$$\sigma^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

- Let us rewrite this as

$$\begin{cases} x = x_1 + K(x_2 - x_1), \\ \sigma^2 = (1 - K)\sigma_1^2, \\ K = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} = \sigma_1^2 \left(\sigma_1^2 + \sigma_2^2\right)^{-1} \end{cases}$$

Simple Example I

- Notice:
  - The equation

$$x = x_1 + K(x_2 - x_1)$$

    is of the form

    best estimate = prediction + Kalman gain × (measurement − prediction)

  - measurement − prediction ≡ **residual**
  - $\sigma^2 < \sigma_1^2, \sigma_2^2$ (more measurements, less error)
  - if $\sigma_1 \to 0$, $x \to x_1$ and vice versa

Simple Example II

- The previous example had no dynamics, but for example, we could have a system where the position of the particle is given as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u + w,$$

  where u is constant and w is a stochastic error (e.g. multiple scattering)
- In this case, if the measurement at t_1 gave $x_1 \pm \sigma_1$, then our prediction at t_2 will be

$$x = x_1 + u(t_2 - t_1)$$
$$\sigma^2 = \sigma_1^2 + \sigma_w^2(t_2 - t_1)$$

- Now given a measurement at t_2 that is $x_2 \pm \sigma_2$, our best estimate is given by the same kind of expression as before
- Notice that both the prediction error and the measurement error are combined to give an error that is smaller than both

General Form

- The evolution of the system is given as

$$x_k = Ax_{k-1} + w_{k-1} \qquad \text{prediction}$$
$$z_k = Hx_k + v_k \qquad \text{measurement with noise}$$

- Assume the stochastic error in the prediction has covariance matrix $(n \times n)$ Q
- Starting from the covariance matrix at $k-1$, the predicted covariance matrix is

$$C_k^- = AC_{k-1}A^T + Q \qquad \text{all } n \times n \text{ matrices}$$

- Assume the measurement has a covariance matrix $(m \times m)$ R
- We would like to combine the a priori prediction x_k^-, the a posteriori measurement z_k, while taking into account the errors Q and R.

General Form

- The *best estimate* for x_k given the measurement z_k should minimize the a posteriori covariance matrix with respect to x_k
- The solution is the same form as in the examples,

$$x = x_k^- + K(z_k - Hx_k^-)$$
$$K = C_k^- H^T (HC_k^- H^T + R)^{-1}$$
$$C_k = (1 - KH)C_k^-$$

(compare to equations on p.6)

- Recall that H is the $m \times n$ matrix that converts the parameters x into z, i.e.,

$$z_k = Hx_k + v_k$$

How This Applies to GlueX

- Kalman filter implemented in class DTrackFitterKalmanSIMD
- The parameters of interest are called S (what we called x above), and is a 5D vector
- The dynamics of the system (A above) is the charged particles bending in the magnetic field
- The stochastic errors (Q above) associated with the update equations are multiple scattering
- We assume we know both A and Q
- The hits in the drift chambers constitute the measurements (z above)

DTrackFitterKalmanSIMD::KalmanForwardCDC

- This function sets the χ^2 and ndof for a given region of the detector
- We want the best estimate of $\vec{S} = \{x, y, t_x, t_y, q/p\}$
- The update equation is given by the 5×5 matrix J
- The multiple scattering covariance matrix is given by Q
- Each measurement is the CDC hits, which gives 1 measurement, so that z is a 1-dimensional vector
- The measurement "covariance matrix" is given by V, where

$$V = \sigma^2_{\mathrm{CDC}}(dm) + v^2_{\mathrm{CDC}}\sigma^2_{t_0},$$
$$dm = v_{\mathrm{CDC}} \times (t_{\mathrm{drift}} - t_{\mathrm{hit}} - t_0)$$

DTrackFitterKalmanSIMD::KalmanForwardCDC

- Putting all this together:

expression	code
$C^{-} = ACA^{T} + Q$	$C = JCJ^{T} + Q$
$x = x^{-} + K(z - x^{-})$	$S = S + K(dm - d)$
$C = (1 - KH)C^{-}$	$C = (1 - KH)C$
$K = C^{-}H(HC^{-}H^{T} + R)^{-1}$	$K = CH(HCH^{T} + V)^{-1}$