# 1 Theory appendix

The scattering amplitude for $\gamma\gamma^* \to \pi^0\pi^0$ is given in terms of the Compton tensor, whose low energy expansion in the Compton scattering channel $\gamma\pi^0 \to \gamma\pi^0$ is given in terms of the electric and magnetic polarizabilities of the $\pi^0$. For the case of interest with one real photon, the Compton tensor is given in by two amplitudes, namely:

$$
\begin{aligned}
T_{\mu\nu} &= -(A(s,t,u) + \frac{1}{4}B(s,t,u))(\frac{1}{2}\, s\, g_{\mu\nu} - k_\nu q_\mu) && (1)\\
&+ \frac{1}{4s}B(s,t,u)((s-q^2)p_{-\mu}p_{-\nu} - 2(k\cdot p_-\ q_\mu p_{-\nu} + q\cdot p_- k_\nu p_{-\mu} - g_{\mu\nu}k\cdot p_-\ \ q\cdot p_-) && (2)
\end{aligned}
$$

Here $s = W_{\pi\pi}^2$ is the invariant mass squared of the two $\pi^0$s, $k$ the momentum of the beam photon, $q$ the momentum of the virtual photon, and $p_-$ the $p_- = p_1 - p_2$ the momentum difference between the two pions.

The limit of interest for the polarizabilities is:

$$
\begin{aligned}
\alpha_\pi &= -\frac{\alpha}{2M_\pi}(A(s,t,u) - \frac{2}{s}M_\pi^2 B(s,t,u))|_{s=0,t=u=M_\pi^2}\\
\beta_\pi &= \frac{\alpha}{2M_\pi}A|_{s=0,t=u=M_\pi^2} && (3)
\end{aligned}
$$

where $\alpha_\pi$ $\beta_\pi$ are the electric and magnetic polarizabilities respectively.

The low energy limit is analyzed in ChPT. At the lowest significant order, i.e., one loop, the $\pi^0$ polarizabilities are entirely given in terms of known quantities, namely:

$$
\alpha_{\pi_0} = -\beta_{\pi_0} = -\frac{\alpha}{96\pi^2 M_\pi F_\pi^2} \simeq -0.55 \times 10^{-4}\ \mathrm{fm}^3 \qquad (4)
$$

The positive magnetic susceptibility indicates that the $\pi_0$ is diamagnetic, and naturally the negative electric polarizability tells that it behaves as a dielectric.

There are higher order corrections in the chiral expansion to the above prediction corresponding to a two-loop calculation, which is undefined up to two low energy constants $h_\pm$ in the notation of Ref. [**?**], expected to be significant for the corrections.

The amplitudes $A$ and $B$ are constrained by unitarity and analiticity to satisfy dispersion relations. In particular below $s \sim 0.8$ GeV$^2$ the dominant contributions are for the pair of pions in an S-wave. The rather well established S-wave phase shifts thus allow for implementing dispersion relations [**?**]. In this proposal the model by Donoghue and Holstein [**?**] for implementing the dispersive representation using S-wave final state interaction was adopted. The model implements twice subtracted dispersion relations for the isospin 0 and 2 components of the amplitude A with the addition of t- and u-channel resonance exchanges for both A and B. The four subtraction constants require the experimental input of the cross section to be measured by the proposed experiment.

A summary of useful theory results is the following:

1) representation of the Compton amplitudes:

$$
\begin{aligned}
s\,A(s,t,u) &= -\frac{2}{3}(f_0(s)-f_2(s)) + \frac{2}{3}(p_0(s)-p_2(s)) - \frac{s}{2}\sum_{V=\rho,\omega} R_V\left(\frac{t+M_\pi^2}{t-M_V^2} + \frac{u+M_\pi^2}{u-M_V^2}\right) \\
B(s,t,u) &= -\frac{1}{8}\sum_{V=\rho,\omega} R_V\left(\frac{1}{t-M_V^2} + \frac{1}{u-M_V^2}\right) \\
R_V &= \frac{6M_V^2}{\alpha}\frac{\Gamma(V\to\pi\gamma)}{(M_V^2-M_\pi^2)^3}
\end{aligned}
\tag{5}
$$

where $V=\rho,\ \omega$,

$$
\begin{aligned}
p_I(s) &= f_I^{\mathrm{Born}}(s) + p_I^A(s) + p_I^\rho(s) + p_I^\omega(s) \\
p_0^A(s) = p_2^A(s) &= \frac{L_9^r + L_{10}^r}{F_\pi^2}\left(s + \frac{M_A^2 - M_\pi^2}{\beta(s)}\log\frac{1+\beta(s)+s_A/s}{1-\beta(s)+s_A/s}\right) \\
p_0^\rho(s) &= \frac{3}{2}R_\rho\left(\frac{M_\rho^2}{\beta(s)}\log\frac{1+\beta(s)+s_\rho/s}{1-\beta(s)+s_\rho/s}\right) \\
p_2^\rho(s) &= 0 \\
p_0^\omega(s) = -\frac{1}{2}p_0^\omega(s) &= -\frac{1}{2}R_\omega\left(\frac{M_\omega^2}{\beta(s)}\log\frac{1+\beta(s)+s_\omega/s}{1-\beta(s)+s_\omega/s} - s\right),
\end{aligned}
\tag{6}
$$

where $\beta(s)=\sqrt{\frac{s-4M_\pi^2}{s}}$, $M_A$ the mass of the $A_1$ resonance. The $f_I$s are given by the dispersive representation:

$$
f_I(s) = p_I(s) + \Omega_I(s)\left(c_I + d_I\,s - \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} p_I(s')\mathrm{Im}(\Omega_I^{-1}(s'))\frac{ds'}{(s'-s)s'^2}\right), \tag{7}
$$

with the Omnès function:

$$
\Omega_I(s>4M_\pi^2) = e^{i\phi_I(s)}\exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\phi_I(s')-\phi_I(s)}{s'-s}\frac{ds'}{s'} + \frac{\phi_I(s)}{\pi}\log\frac{4M_\pi^2}{s-4M_\pi^2}\right). \tag{8}
$$

the phases $\phi_I$ are related to the corresponding $\pi\pi$ S-wave phase shifts according to:

$$
\begin{aligned}
\phi_0(s) &= \theta(M-\sqrt{s})\delta_0^0(s) + \theta(\sqrt{s}-M)(\pi-\delta_0^0(s)) \\
\phi_2(s) &= \delta_0^2(s),
\end{aligned}
\tag{9}
$$

where $M$ is the mass of the $f_0$ resonance.

The values used for the parameters entering the representations above are:

$$
\begin{aligned}
L_9^r + L_{10}^r &= 1.43 \pm 0.27 \times 10^{-3} \\
s_i &= 2(M_i^2 - M_\pi^2) \\
R_\omega &= 1.35/GeV^2; \quad R_\rho = 0.12/GeV^2
\end{aligned}
\tag{10}
$$

and the $\pi\pi$ phase shifts are well approximated up to $\sqrt{s} \sim 1.5 GeV$ by the parametrization:

$$\delta_0^I(s) = \arcsin\left(\frac{\Gamma_I}{2\sqrt{(\sqrt{s}-M_I)^2+\frac{\Gamma_I^2}{4}}}\right) + \sum_{n=0}^{N} a_n \ (\sqrt{s})^n \qquad (11)$$

where we include one single resonance for each $I = 0, 2$.

For the available data we need only up to $N = 3$ for $I = 0$, with the result:

$$\begin{aligned} M_0 = 0.994 GeV; \quad & \Gamma_0 = 0.0624 GeV \\ a_0 = -1.439; \quad & a_1 = 6.461/GeV; \quad a_2 = -5.529/GeV^2; \quad a_3 = 2.022/GeV^3 \end{aligned} \qquad (12)$$

For the case $I = 2$ one finds that the resonance term is not needed at all and a good fit is provided with $N = 3$ with the result:

$$a_0 = -0.878; a_1 = -0.611/GeV; \quad a_2 = -0.083/GeV^2; \quad a_3 = 0.115/GeV^3 (13)$$

The $\gamma\gamma \to \pi_0\pi_0$ in the S-wave approximation valid up to about $\sqrt{s} \sim 0.9 GeV$ is given by:

$$\begin{aligned} \sigma_{\gamma\gamma\to\pi^0\pi^0}(|\cos\theta| < Z)(s) &= \frac{\pi\alpha_{EM}^2}{s^2}\frac{Z}{2}\sqrt{s(s-4M_\pi^2)} \qquad (14) \\ &\times (| A(s)s - M_\pi^2 B(s) |^2 \\ &+ \frac{1}{s^2}\left(M_\pi^4 - \frac{1}{16}(\frac{Z^2}{3}s(4M_\pi^2 - s) + 4(s-2M_\pi^2)^2)\right) | B(s) |^2) \end{aligned}$$

Fitting to the Cristal Ball data the parameters $c_0,\ d_0,\ c_2,\ d_2$ can be estimated, giving in the corresponding units:

$$\begin{aligned} c_0 &= -0.529 \\ d_0 &= -2.033 \\ c_2 &= 0.953 \\ d_2 &= -1.271. \end{aligned} \qquad (15)$$